RUSSIAN RESEARCH CENTRE РОССИЙСКИЙ НАУЧНЫЙ ЦЕНТР
KURCHATOV INSTITUTE "КУРЧАТОВСКИЙ ИНСТИТУТ"

Magnetic Coupling of the Toroidal Plasma with External Asymmetries

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THEORY OF FUSION PLASMAS
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➢ Coupling:

Magnetic interaction of the plasma with the wall (vacuum vessel), correction coils, error field

What will be the plasma reaction to the applied magnetic perturbation?

The Problem

Toroidal Geometry:

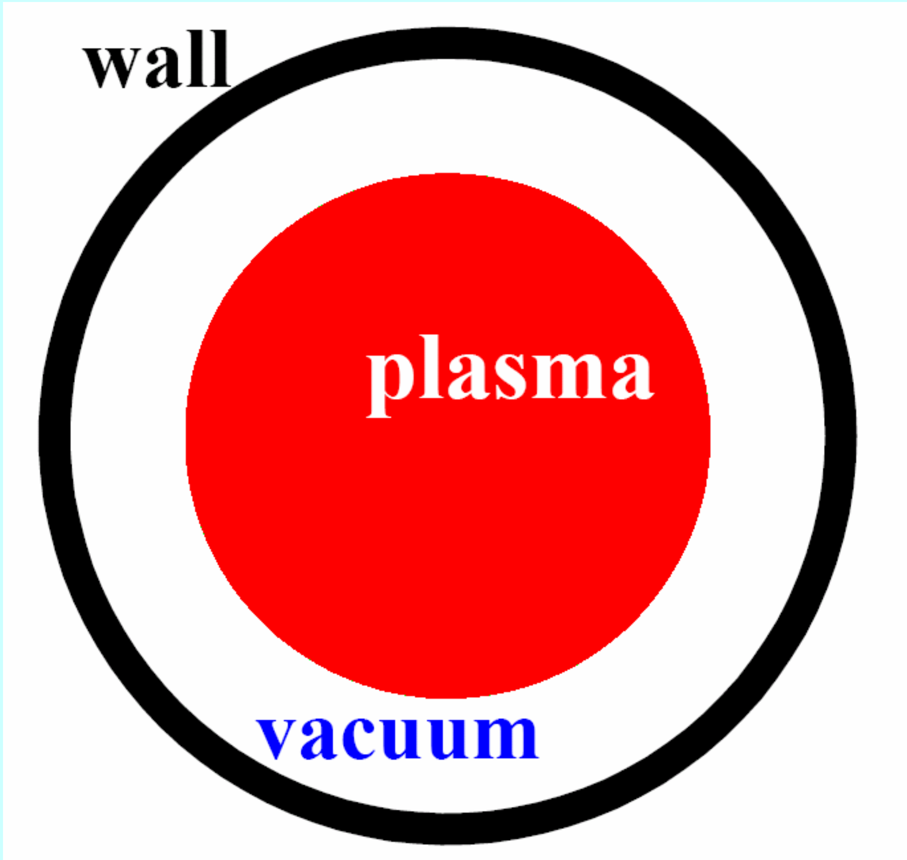
Plasma – vacuum gap – wall – vacuum – …

Approximation:

sometimes – thin-wall

The goal:

Coupling of the inner and outer solutions for the magnetic perturbation



Existing approaches

Different areas (plasma, wall, vacuum) – Different solvers

Examples: MARS (MHD) + CARIDDI (wall currents)

DCON (MHD) + VALEN (wall currents)

The coupling/decoupling is also needed in IPEC (the Ideal Perturbed Equilibrium Code)

DCON (MHD) + coupling of ξ to external field (vacuum)

We derive the equation covering the "coupling strategy" described in

F. Villone, R. Albanese, Y. Q. Liu, A. Portone and G. Rubinacci, **3D effects of conducting structures on RWMs control in ITER** *34th EPS Conf. on Plasma Physics*, Warsaw, 2007, P-5.125.

R. Albanese, Y. Q. Liu, A. Portone, G. Rubinacci, F. Villone, **Coupling Between a 3-D Integral Eddy Current Formulation and a Linearized MHD Model for the Analysis of Resistive Wall Modes** *IEEE Transaction on Magnetics* **44**, 1654 (2008)

A. Portone, F. Villone, Y. Q. Liu, R. Albanese, and G. Rubinacci, **Linearly perturbed MHD equilibria and 3D eddy current coupling via the control surface method** *Plasma Phys. Control. Fusion* **50**, 085004 (2008)

This also solves a similar coupling problem from

Jong-kyu Park, A. H. Boozer, A. H. Glasser, **Computation of three-dimensional tokamak and spherical torus equilibria**, Phys. Plasmas **14**, 052110 (2007)

PHYSICS OF PLASMAS **14**, 052110 (2007)

Computation of three-dimensional tokamak and spherical torus equilibria

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the primary interest is in the response of the plasma to very small perturbations, i.e., $b/B \approx 10^{-2}$ to 10^{-4}, which can be calculated using the theory of perturbed equilibria.

The ideal plasma response to external magnetic perturbation can be **computed** with high accuracy by the **code** that constructs a **relevant interface between the actual field and the external field on the control surface.**

PHYSICS OF PLASMAS 14, 052110 (2007)

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Solution scheme: **Actual field** $\Rightarrow$ **external surface current**

$\Rightarrow$ **required external field**

Linear operators: $\vec{b}\cdot\hat{n}=\hat{\Lambda}[\vec{K}^x]$ **and** $\vec{b}^x\cdot\hat{n}=\hat{L}[\vec{K}^x]$

finally $\vec{b}\cdot\hat{n}=\hat{P}[\vec{b}^x\cdot\hat{n}]$ **with** $\hat{P}=\hat{\Lambda}\hat{L}^{-1}$.

*34th EPS Conference on Plasma Phys. Warsaw, 2 - 6 July 2007 ECA Vol.**31F**, P-5.125 (2007)*

3D effects of conducting structures on RWMs control in ITER

F. Villone[1], R. Albanese[2], Y.Q. Liu[3], A. Portone[4], G. Rubinacci[5]

The CarMa code, a self-consistent coupling between the MHD code MARS-F and the 3D eddy currents code CARIDDI, is applied to ITER geometry for the evaluation of the effects of 3D conducting structures on Resistive Wall Modes (RWM) control.

Coupling strategy: the CarMa code

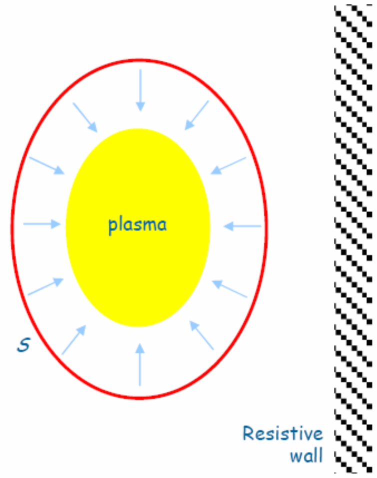
Main assumption: **plasma mass is neglected**

- good approximation if the time scale is much longer than Alfvén time (related to plasma mass)
- plasma response to given input is instantaneous

A (de-)**coupling surface** *S* is chosen

- any surface in between plasma and conducting structures
- plasma-wall interaction is decoupled via S

The plasma (instantaneous) response to a given magnetic flux density perturbation on S is computed as a plasma response matrix.



1654 IEEE TRANSACTIONS ON MAGNETICS, VOL. 44, NO. 6, JUNE 2008

Coupling Between a 3-D Integral Eddy Current Formulation and a Linearized MHD Model for the Analysis of Resistive Wall Modes

R. Albanese[1], Y. Q. Liu[2], A. Portone[3], G. Rubinacci[1], and F. Villone[4]

Coupling strategy

A surface is chosen, in between the plasma and the conducting structures

The plasma response to a given magnetic perturbation is computed as a plasma response matrix

The effect of 3D structures on plasma is evaluated by computing the magnetic flux density on S due to 3D currents

The currents induced in the 3D structures by plasma are computed via an *equivalent surface current* distribution on S

➢ **In the both cases the needed relations are:**

$$\vec{b}\cdot\hat{n} = \hat{P}[\vec{b}^x\cdot\hat{n}]$$

Jong-kyu Park, et al., Phys. Plasmas **14**, 052110 (2007)

$$\mathbf{b}_N = \mathbf{W}\mathbf{b}_N^{ex}$$

A. Portone, et al., Plasma Phys. Control. Fusion **50**, 085004 (2008)

➢ **In the both cases not only** $\mathbf{b}\cdot\mathbf{n}$**, but** $\mathbf{b}\times\mathbf{n}$ **on** S **is also known**

"IPEC uses the displacement of the plasma boundary to determine a part of the perturbed magnetic field that is normal to the unperturbed plasma boundary and a **part that is tangential to the plasma boundary**" (Jong-kyu Park, et al., 2007)

"In particular, each time we find the **tangential** component of ***b*** on S_e" (A. Portone, et al., 2008)

➢ **In the both cases to solve the problem an additional** ("external, equivalent, superficial") **surface current is introduced and calculated**

This step is actually NOT necessary

Starting equations

Plasma:

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \mathbf{j} \times \mathbf{B} + \mathbf{f}$$

Wall:

$$\mathbf{j} = \sigma \mathbf{E}$$

All space:

$$\mathrm{rot}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\mathrm{div}\,\mathbf{B} = 0,$$

$$\mathrm{rot}\,\mathbf{B} = \mu \mathbf{j}$$

(μ : permeability)

Outside the plasma:

vacuum gap,

wall ($\mu = \mathrm{const}$, $\sigma = \mathrm{const}$ **)**

correction coils, error field

boundary conditions: $\langle \mathbf{n} \cdot \mathbf{B} \rangle = 0$ $\quad \langle \mathbf{n} \times \mathbf{B} / \mu \rangle = 0$

Definitions

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{b}(r, \zeta, z)$$

Magnetic field: Unperturbed + perturbation

$$\mathbf{b} = \mathbf{b}^{plasma} + \mathbf{b}^{wall} + \mathbf{b}^{external}$$

In vacuum $\mathbf{b} = \nabla\varphi$ **with** $\nabla^2\varphi = 0$

$$\varphi = \varphi^{plasma} + \varphi^{wall} + \varphi^{external}$$

with $\nabla^2\varphi^i = 0$ **in different areas for different** φ^i

Formulation of the problem

Assume that equation

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \mathbf{j} \times \mathbf{B} + \mathbf{f}$$

is solved with given disturbed boundary (inside a torus S)

A relation between the total field and the external field is needed on the 'control surface' S

$$\mathbf{B}^{out}(\mathbf{B}) = ?$$

Here

$$\mathbf{B} = \mathbf{B}^{out} + \mathbf{B}^{pl}.$$

General solution

Use Biot–Savart law

$$\mathbf{B}^{pl} = \frac{\mu_0}{4\pi}\nabla\times\int_{plasma}\frac{\mathbf{j}(\mathbf{r}_{pl})}{\left|\mathbf{r}-\mathbf{r}_{pl}\right|}dV_{pl} = \frac{\mu_0}{4\pi}\int_{plasma}\mathbf{j}(\mathbf{r}_{pl})\times\frac{\mathbf{r}-\mathbf{r}_{pl}}{\left|\mathbf{r}-\mathbf{r}_{pl}\right|^3}dV_{pl},$$

integration over the plasma volume.

Here $\mathbf{j}=\nabla\times\mathbf{B}/\mu_0$ **must be known from the solution of the perturbed equilibrium problem**

Can it be transformed into the surface integral over the control surface (in general, not the plasma surface)?

Volume integral → control surface

1. Use the identity

$$\mu_0 \int_V (\mathbf{j} \times \nabla f)\, dV = \int_S \{(\mathbf{n} \times \mathbf{B}) \times \nabla f + \nabla f (\mathbf{n} \cdot \mathbf{B})\} dS - \int_V \mathbf{B} \nabla^2 f\, dV,$$

where $\mathbf{n}$ is the unit outward normal to S.

2. With $f = (4\pi |\mathbf{r} - \mathbf{r}'|)^{-1}$, which satisfies $\nabla^2 f(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}')$:

$$\mathbf{B}^{pl}(\mathbf{r}) = \frac{1}{4\pi} \int_S \left\{ (\mathbf{n} \times \mathbf{B}) \times \frac{\mathbf{r} - \mathbf{r}_s}{|\mathbf{r} - \mathbf{r}_s|^3} + (\mathbf{n} \cdot \mathbf{B}) \frac{\mathbf{r} - \mathbf{r}_s}{|\mathbf{r} - \mathbf{r}_s|^3} \right\} dS + \nu \mathbf{B}(\mathbf{r})$$

where $$\nu = \int_V \delta(\mathbf{r} - \mathbf{r}')\, dV' = \begin{cases} 1 & if \quad \mathbf{r} \in V \\ 0.5 & if \quad \mathbf{r} \in S \\ 0 & if \quad \mathbf{r} \notin V \cup S \end{cases}$$

Maxwell equations are linear, $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$, then

$$\mathbf{b}^{out} = \mathbf{b} - \mathbf{b}^{pl} = \mathbf{b}(1-\nu) - \frac{1}{4\pi}\int_S \left\{ (\mathbf{n}\times\mathbf{b})\times\frac{\mathbf{r}-\mathbf{r}_s}{|\mathbf{r}-\mathbf{r}_s|^3} + (\mathbf{n}\cdot\mathbf{b})\frac{\mathbf{r}-\mathbf{r}_s}{|\mathbf{r}-\mathbf{r}_s|^3} \right\} dS$$

Or, outside the *control surface* S:

$$\mathbf{b}^{pl}(\mathbf{r}) = \frac{1}{4\pi}\int_S \left\{ (\mathbf{n}\times\mathbf{b})\times\frac{\mathbf{r}-\mathbf{r}_s}{|\mathbf{r}-\mathbf{r}_s|^3} + (\mathbf{n}\cdot\mathbf{b})\frac{\mathbf{r}-\mathbf{r}_s}{|\mathbf{r}-\mathbf{r}_s|^3} \right\} dS$$

Can be used to find $\mathbf{b}^{pl}$ at the wall position

Compare to the CarMa coupling scheme

1. For some *vacuum* magnetic field $\mathbf{b}_v$ with $\mathbf{n}\cdot\mathbf{b}_v = \mathbf{n}\cdot\mathbf{b}$ at S:

$$0 = \frac{1}{4\pi}\int_S \left\{ (\mathbf{n}\times\mathbf{b}_v)\times\frac{\mathbf{r}-\mathbf{r}_s}{|\mathbf{r}-\mathbf{r}_s|^3} + (\mathbf{n}\cdot\mathbf{b}_v)\frac{\mathbf{r}-\mathbf{r}_s}{|\mathbf{r}-\mathbf{r}_s|^3} \right\} dS + \nu\mathbf{b}_v \, .$$

2. Subtract from the equation for $\mathbf{b}^{pl}$, get the CarMa result:

$$\mathbf{b}^{pl}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_S \mathbf{i}\times\frac{\mathbf{r}-\mathbf{r}_s}{|\mathbf{r}-\mathbf{r}_s|^3} dS + \nu(\mathbf{b}-\mathbf{b}_v) \, ,$$

where

$$\mu_0\mathbf{i} = \mathbf{n}\times(\mathbf{b}-\mathbf{b}_v)$$

An alternative approach (1)

1. Start from $\nabla^2 \varphi^i = 0$,

where φ^i **describe the contributions to** $\varphi = \varphi^{plasma} + \varphi^{wall} + \varphi^{external}$

2. Use the Green's second identity with $g = (4\pi|\mathbf{r} - \mathbf{r}'|)^{-1}$, **which satisfies** $\nabla^2 g(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}')$, **to get**

$$-\varphi^i(\mathbf{r})\int_V \delta(\mathbf{r} - \mathbf{r}')dV' = \int_S (\varphi^i \nabla g - g\nabla \varphi^i) \cdot d\mathbf{S}$$

3. Apply this for each φ^i **with *different* regions depending on** φ^i.

Twice contrary to: M. S. Chance, *Phys. Plasmas* **4**, 2161 (1997).

An alternative approach (2)

1. Main equation in M. S. Chance, *Phys. Plasmas* **4**, 2161 (1997):

$$\varphi(\mathbf{r}) = \int_{control} (\varphi \nabla g - g \nabla \varphi) \cdot d\mathbf{S}_c - \int_{wall} (\varphi \nabla g - g \nabla \varphi) \cdot d\mathbf{S}_w$$

2. Our first separate equation:

$$\varphi^{plasma}(\mathbf{r}) = \int_{control} (\varphi \nabla g - g \nabla \varphi) \cdot d\mathbf{S}_c$$

3. Our second separate equation:

$$\varphi^{w}(\mathbf{r}) + \varphi^{ext}(\mathbf{r}) = -\int_{wall} (\varphi \nabla g - g \nabla \varphi) \cdot d\mathbf{S}_w$$

Compare

$$\mathbf{b}^{plasma}(\mathbf{r}) = \frac{1}{4\pi}\int_S \left\{(\mathbf{n}\times\mathbf{b})\times\frac{\mathbf{r}-\mathbf{r}_s}{\left|\mathbf{r}-\mathbf{r}_s\right|^3} + (\mathbf{n}\cdot\mathbf{b})\frac{\mathbf{r}-\mathbf{r}_s}{\left|\mathbf{r}-\mathbf{r}_s\right|^3}\right\} dS$$

and

$$\varphi^{plasma}(\mathbf{r}) = \int_{control} (\varphi\nabla g - g\nabla\varphi)\cdot d\mathbf{S}_c$$

It can be shown that

$$\nabla\varphi^{plasma} = \int_{control} \{(\mathbf{n}_c\times\nabla\varphi)\times\nabla g + (\mathbf{n}_c\cdot\nabla\varphi)\nabla g\} dS_c$$

Thin-wall equations (1)

$$\mathbf{j} = \sigma\mathbf{E}, \quad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \quad \Rightarrow \quad |\nabla a|\frac{\partial}{\partial t}\mathbf{b}\cdot\mathbf{n}_w = \nabla\cdot(\nabla a\times\mathbf{j}/\sigma)$$

where $a(\mathbf{r}) = \mathrm{const}$ **describes the toroidal surfaces:**

$a = a_w$**:** **the inner side of the wall (** S_w **)**

$a = a_w + da$**:** **the inner outer side of the wall (** S_{w+} **)**

Assume that the wall is also thin *magnetically*:

$$\mathbf{n}_w\cdot\nabla\varphi_-^w = \mathbf{n}_w\cdot\nabla\varphi_+^w \quad \textbf{Then} \quad \mathbf{j} = \nabla\kappa\times\nabla H$$

where $H(a-a_w) = H_w$ **is the Heaviside step function**

Thin-wall equations (2)

$$|\nabla a|\frac{\partial}{\partial t}\mathbf{b}\cdot\mathbf{n}_w = \nabla\cdot(\nabla a\times\mathbf{j}/\sigma) \quad\Rightarrow\quad \tau_w\frac{\partial}{\partial t}\mathbf{b}\cdot\mathbf{n}_w = \hat{l}\mu_0\kappa$$

where $\tau_w = \mu_0\sigma_0 r_0 d_0$ is the 'wall time' with

σ_0, r_0 and d_0 the constants representing 'in average' the conductivity, minor radius and thickness of the wall,

and $\hat{l}$ is the operator defined on S_w by

$$\hat{l}X \equiv r_0\frac{d_w}{d_0}\nabla\cdot\frac{\sigma_0 d_0^2}{\sigma d_w^2}[\mathbf{n}_w\times(\nabla X\times\mathbf{n}_w)]$$

Thin-wall equations (3)

$$\mathbf{j} = \nabla\kappa \times \nabla H \quad \Rightarrow \quad \mathbf{b}^w = \frac{\mu_0}{4\pi} \int_{wall} (\nabla\kappa \times \mathbf{n}_w) \times \frac{\mathbf{r} - \mathbf{r}_w}{|\mathbf{r} - \mathbf{r}_w|^3} dS_w$$

Also, from $\mathbf{j} = \nabla \times (\mathbf{B}/\mu) = \nabla \times (\kappa \nabla H)$ **we obtain**

$\varphi_-^w - \varphi_+^w = \mu_0 \kappa$, **and, after simple transformations,**

$$\varphi^w = -\mu_0 \int_{wall} \kappa \nabla g \cdot d\mathbf{S}_w = -\tau_w \frac{\partial}{\partial t} \int_{wall} \hat{l}^{-1} (\mathbf{b} \cdot \mathbf{n}_w) \nabla g \cdot d\mathbf{S}_w$$

Both representations are equivalent: $\nabla \varphi^w = \mathbf{b}^w$

Approximations and applications (1)

In the cylindrical coordinates (R, ζ, z) **with toroidal angle** ζ **:**

$$g \equiv \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} = \frac{1}{4\pi^2\sqrt{RR'}}\sum_{n=-\infty}^{\infty} e^{in(\zeta-\zeta')} Q_{n-1/2}(\chi),$$

$$Q_{n-1/2}(\chi) \equiv \frac{1}{2\sqrt{2}}\int_0^{2\pi}\frac{\cos nu}{\sqrt{\chi-\cos u}}du$$

is the half-integer degree Legendre function of the second kind, $\chi \equiv \frac{2}{k^2} - 1$ **with** $k^2 \equiv \frac{4RR'}{(R+R')^2+(z-z')^2}$

$$Q_{n-1/2}(\chi) = i_n(k) + \ln\frac{k}{\sqrt{1-k^2}}$$

where $i_n(k)$ **is the is a function finite at** $k^2 = 1$**.**

Approximations and applications (2)

Axial symmetry of the integration surfaces,
large-aspect ratio,
circular plasma and wall $\Rightarrow$

$$\tau_w \frac{\partial B_m}{\partial t} = \Gamma_m B_m - \Gamma_m^0 B_m^{ext}$$

which is the result of 'cylindrical' theory.

Cylindrical approximation: $b_r = \sum b_m(r,t)\exp i(m\theta - n\zeta)$

$$b_m(r_w) = B_m = B_m^{pl} + B_m^{wall} + B_m^{ext},$$

$\Gamma_m = -2m\dfrac{B_m^{wall} + B_m^{ext}}{B_m}$ **also found from** $\dfrac{rb_m'}{b_m} = -(\mu+1) - \dfrac{2\mu\Gamma_m x^{2\mu}}{2\mu + \Gamma_m(1 - x^{2\mu})}$

Main equation

Equation for the mode amplitude at the wall:

$$\tau_w \frac{\partial B_m}{\partial t} = \Gamma_m B_m + 2m B_m^{ext}$$

When $B_m^{ext} = 0$

$$\Gamma_m = \tau_w \left(\gamma_0 + in\Omega_0 \right),$$

γ_0 **is the natural growth/decay rate,**

Ω_0 **is the natural toroidal rotation frequency of the mode**

Resonant Field Amplification

$$\tau_w \frac{\partial B_m}{\partial t} = \Gamma_m B_m + 2mB_m^{ext}$$

At $\Gamma_m = \text{const}$, $\text{Re}\,\Gamma_m < 0$, $B_m^{ext} = \text{const}$ **(static perturbation)**

$$B_m^{st} = AB_m^{ext}$$ **with** $$A = -\frac{2m}{\Gamma_m}$$

General geometry: $\varphi = \varphi^{out} + \varphi^{pl}$,

Given φ^{out}**, find** φ^{pl}**, calculate the 'amplification' ratio**

$$\varphi / \varphi^{out} = 1 + \varphi^{pl} / \varphi^{out}.$$

Summary

- **The mentioned problems are resolved analytically by an explicit expression. The general solution gives the necessary output at the same input as in CarMa.**

- **This shows, in particular, that a part of the CarMa numerical procedure is redundant and can be replaced by a precise analytical solution.**

- **This solves the Coupling problem as needed in the IPEC formulation** (Park, et al., Phys. Plasmas **14**, 052110 (2007))**.**

- **The solution is ready for use, will be useful for any RWM code and in the "Perturbed equilibrium" problems.**

For more details see

V.D. Pustovitov, **General formulation of the resistive wall mode coupling equations,** Phys. Plasmas **47**, 072501 (2008);

V.D. Pustovitov, **Decoupling in the problem of tokamak plasma response to asymmetric magnetic perturbations,** Plasma Phys. Control. Fusion **50,** 105001 (2008).