Zonal flows in stellarator geometry

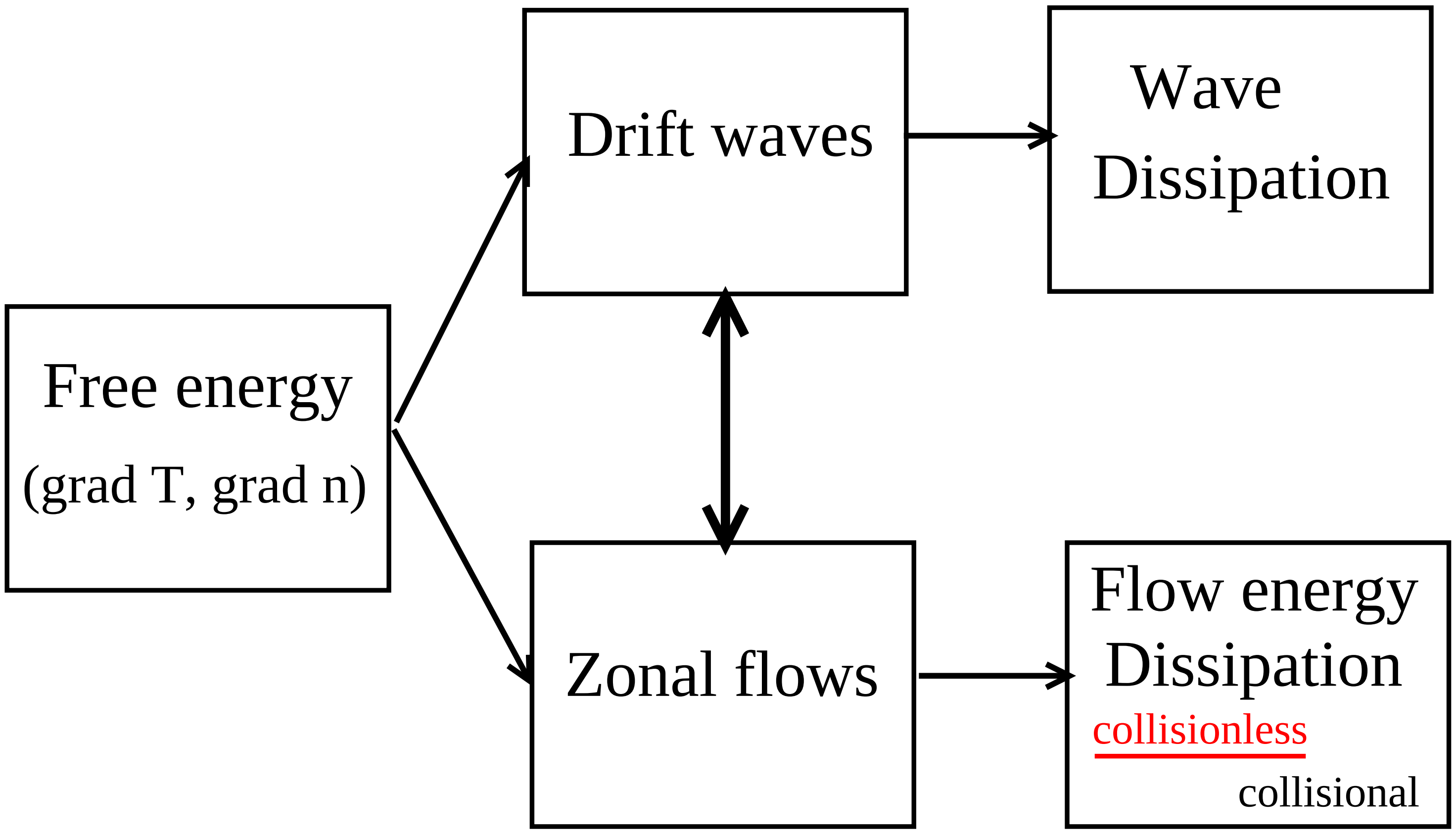
Alexey Mishchenko, Per Helander, Axel Könies

Keywords: zonal flow, collisionless damping, Rosenbluth-Hinton theory, stellarator, locally-trapped particles, bounce-averaged radial drifts.

Acknowledgments: H. Sugama, R. Kleiber, J. Geiger, J. Nührenberg

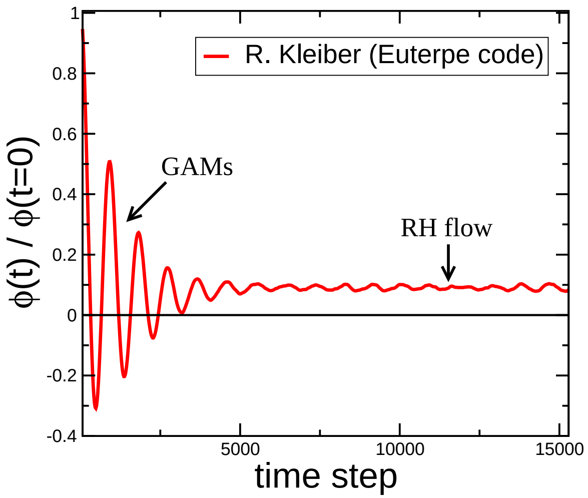
QUALITATIVE PICTURE

Drift wave - Zonal flow paradigm



Rosenbluth-Hinton theory

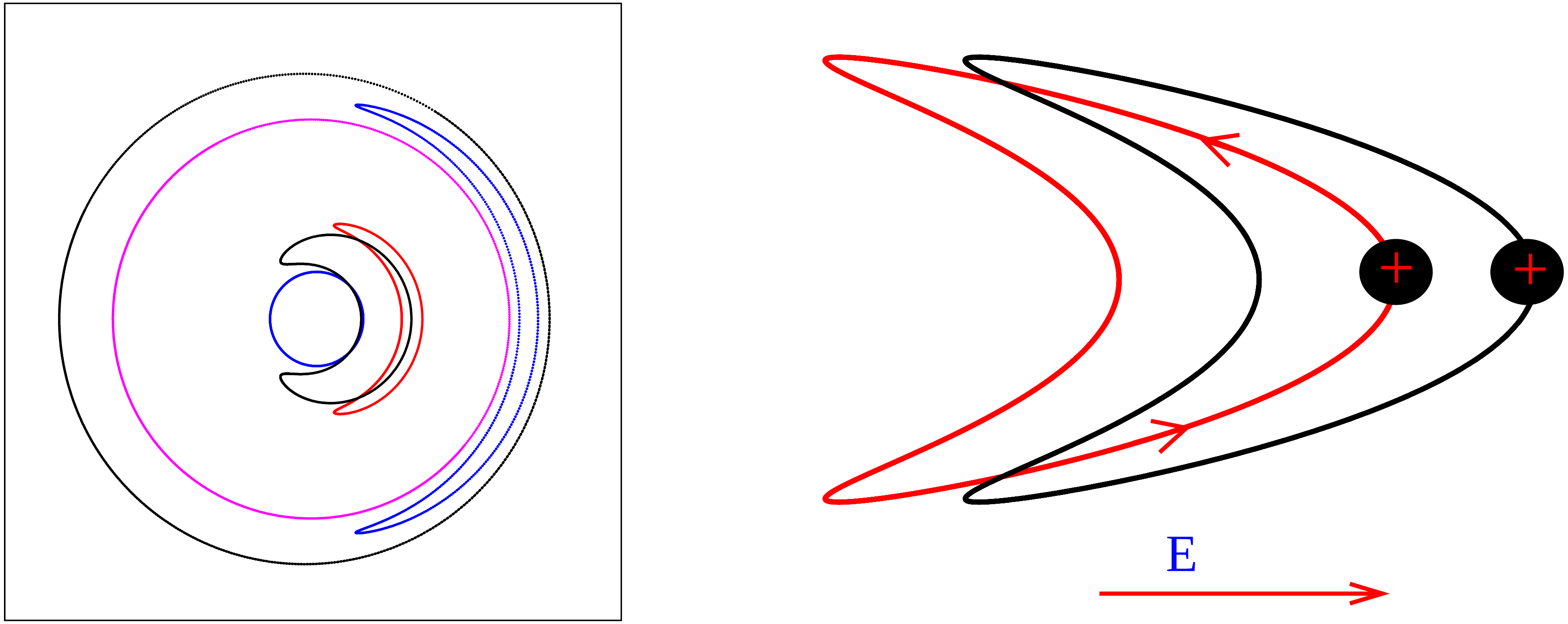
Initial density perturbation $n(t = 0)$ → **Plasma** Linearized equations → Electrostatic potential $\phi(t)$?



Rosenbluth-Hinton flow

$$\frac{\phi(t)}{\phi(t=0)} = \frac{1}{1 + 1.6q^2/\sqrt{\varepsilon}}$$

Tokamak particle orbits and the electric field

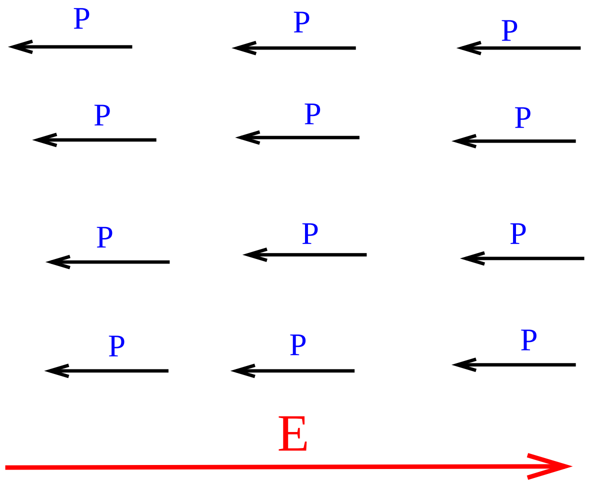


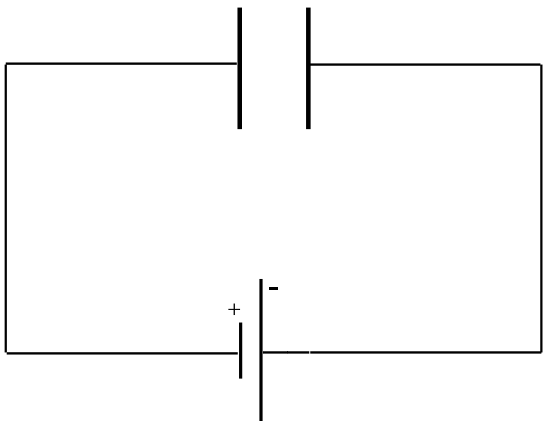
Electrons are “frozen” to the magnetic field line

Shielding due to polarization

Shift of the ion gyro-orbits
$\Longrightarrow$ classical polarization

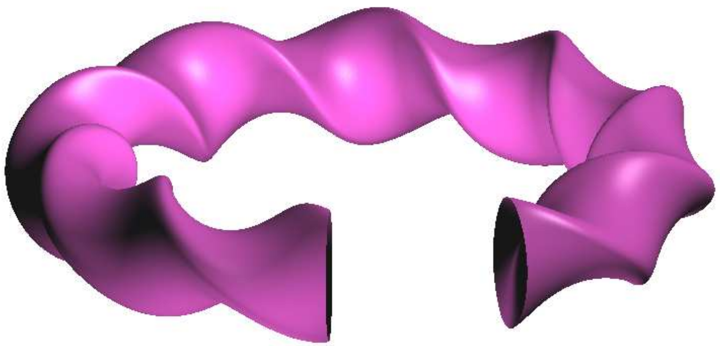
Shift of the ion banana orbits
$\Longrightarrow$ neoclassical polarization

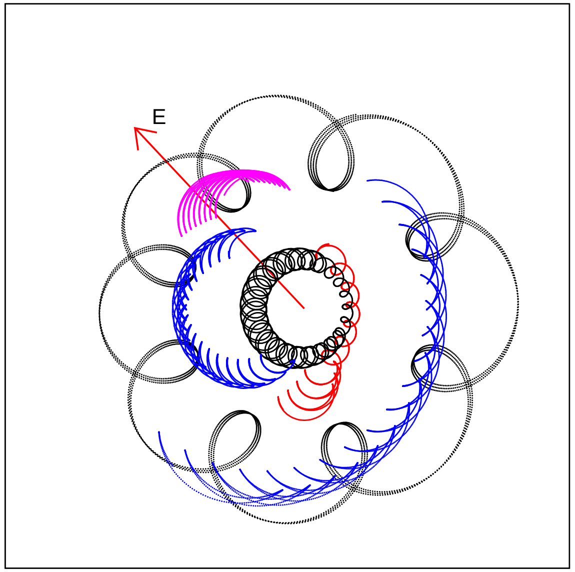


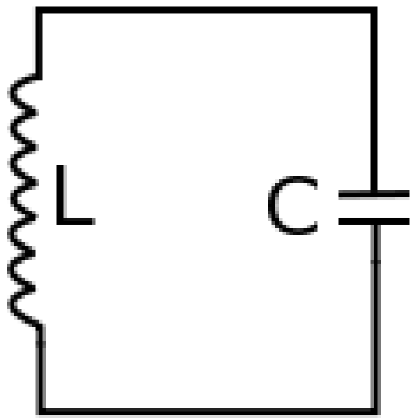


The plasma acts as a capacitor, and the radial voltage is related to the current as $i(t) = C \, \mathrm{d}U/dt$

Stellarator particle orbits and the electric field







A net current arises proportional to the time integral of the voltage
$i(t) = L^{-1} \int_0^t U(t')dt' + C\mathrm{d}U/\mathrm{d}t$.
The plasma behaves like an LC-circuit and oscillates at a frequency $\Omega = (LC)^{-1/2}$.

DERIVATION

Basic equations

Start from the gyrokinetic system of equations:

$$\frac{\partial f_{a1}}{\partial t} + v_{||}\nabla_{||} f_{a1} + \vec{v}_d \cdot \nabla f_{a1} = -\dot{\varepsilon}\frac{\partial f_{a0}}{\partial \varepsilon}$$

$$-\sum_{b=i,Z} \nabla \cdot \left(\frac{e_b^2 n_{0b}}{T_b} \rho_b^2 \nabla_\perp \phi \right) = \sum_{a=i,Z,e} e_a n_a$$

The kinetic energy $\varepsilon = mv^2/2$ changes
(due to the electrostatic field perturbation) $\dot{\varepsilon} = -e\,\vec{v}_d \cdot \nabla \phi$

Here, $\vec{v}_d = \rho_{||} \nabla \times \vec{v_{||}}$ and $\rho_{||} = v_{||}/\omega_c$.

The electrostatic potential depends on radial coordinate only $\phi = \phi(s,t)$.

Basic equations

From the gyrokinetic equation directly follows the continuity equation:

$$\frac{\partial \langle n_a \rangle}{\partial t} + \left\langle \int \mathrm{d}^3 v \, \vec{v}_{\mathrm{d}a} \cdot \nabla f_{a1} \right\rangle = 0$$

We solve this equation employing the initial condition:

$$-\frac{1}{V'}\frac{\mathrm{d}}{\mathrm{d}s}\left[V' \sum_{b=i,Z} \left\langle \rho_b^2 |\nabla s|^2 \right\rangle \frac{n_{0b} e_b^2}{T_b} \phi'(t=0) \right] = \sum_{a=i,Z,e} e_a \left\langle n_a(t=0) \right\rangle$$

Initial density perturbation is immediately (on a gyro-scale) shielded by classical polarization.

Basic equations

The particle flux in the continuity equation is given by the expression:

$$\left\langle \int \mathrm{d}^3 v \, \vec{v}_d \cdot \nabla f_{a1} \right\rangle = \frac{1}{V'} \frac{\mathrm{d}}{\mathrm{d}s} V' \pi \sum_{\sigma} \int \left\langle \Theta(\xi^2) \left[-\frac{BG}{v_{\parallel}} v_{\parallel} \nabla_{\parallel} f_{a1} + \right.\right.$$

$$\left.\left. + \frac{B f_{a1}}{v_{\parallel}} \overline{\omega}_r \right] \right\rangle \sigma v^3 \mathrm{d}v \mathrm{d}\lambda$$

Here, $\omega_r = \vec{v_d} \cdot \nabla s$ is the radial drift velocity and $v_{\parallel} \nabla_{\parallel} G = \omega_r - \overline{\omega}_r$.

The distribution function is found from the Vlasov equation (neglecting FOW):

$$\frac{\partial f_{a1}}{\partial t} + v_{\parallel} \nabla_{\parallel} f_{a1} = -\frac{e_a \phi'}{T_a} f_{a0} \left(\vec{v_d} \cdot \nabla s \right)$$

with the initial condition employed: $f_{a1}(t = 0) = 0$

Solution

We split the distribution function as follows:

$$f_{a1} = h - f_{a0}\hat{\phi}' G, \quad \hat{\phi}' = \frac{e_a \phi'}{T_a} .$$

Apply the Laplace transform to the kinetic equation:

$$p\mathcal{H} + v_{\|}\nabla_{\|}\mathcal{H} = f_{a0}\hat{\Phi}'\psi \ , \quad \psi = pG - \overline{\omega}_r \ .$$

Solve the kinetic equation by successive approximations assuming $\omega \ll \omega_b$

$$v_{\|}\nabla_{\|}\mathcal{H}^{(0)} = 0 \ , \quad p\mathcal{H}^{(0)} = f_{a0}\hat{\Phi}'\overline{\psi} \ , \quad v_{\|}\nabla_{\|}\mathcal{H}^{(1)} = f_{a0}\hat{\Phi}'\widetilde{\psi} \ .$$

Solution

Apply the Laplace transform to the continuity equation

$$p\mathcal{N}_a(p) - \left\langle n_a(t=0)\right\rangle + \frac{1}{V'}\frac{\mathrm{d}}{\mathrm{d}s}[V'\Gamma_a(p)] = 0 \; .$$

For passing particles, the flux is

$$\Gamma_{a,pass}(p) = -2\pi \int v^3 \mathrm{d}v \mathrm{d}\lambda f_{a0} \frac{e_a\Phi'}{T_a} p \int \mathrm{d}\lambda \left[\left\langle \frac{B}{|v_{\|}|} G^2 \right\rangle - \left\langle \frac{B}{|v_{\|}|} \right\rangle^{-1} \left\langle \frac{B}{|v_{\|}|} G \right\rangle^2 \right] .$$

For trapped particles, the flux is

$$\Gamma_{a,trap}(p) = -\frac{2\pi F_P'}{V'} \frac{e_a\Phi'}{T_a} \int \mathrm{d}v \mathrm{d}\lambda f_{a0} v^3 \oint \mathrm{d}\alpha \sum_n \left(\frac{p\hat{\tau}_b}{2} \left[\overline{G^2} + \frac{\overline{\omega}_r^2}{p^2} \right] \right)_n .$$

Residual zonal flow in stellarator geometry

Quasineutrality equation + inverse Laplace transform give

$$\frac{\phi'(t)}{\phi'(t=0)} = \left(1+\Lambda_1/\Lambda_0\right)^{-1}\cos(\Omega t)\ ,\quad \Omega = \sqrt{\frac{\Lambda_2}{\Lambda_0+\Lambda_1}}\,.$$

Here, the classical polarization and the neoclassical polarizations are

$$\Lambda_0 = \sum_{b=i,Z} \frac{n_{0b}e_b^2}{T_b}\left\langle \rho_b^2 |\nabla s|^2 \right\rangle$$

Compute numerically!

$$\Lambda_1 = \frac{3}{2}\sum_{a=i,Z,e}\frac{n_{0a}e_a^2}{T_a}\int \mathrm{d}\lambda \left[\underbrace{\left(\left\langle \frac{B}{\xi}G_{\mathrm{th}}^2\right\rangle - \left\langle\frac{B}{\xi}\right\rangle^{-1}\left\langle\frac{B}{\xi}G_{\mathrm{th}}\right\rangle^2\right)}_{passing\ particles} + \underbrace{\frac{F_P' v_{\mathrm{th}a}}{V'}\oint \mathrm{d}\alpha \sum_{n=t_1,t_2,\ldots}\left(\frac{\hat{\tau}_{b\mathrm{th}}}{2}\overline{G_{\mathrm{th}}^2}\right)_{n,a}}_{trapped\ particles}\right]$$

The frequency is related to the bounce-averaged radial drifts through

$$\Lambda_2 = \underbrace{\frac{15}{2}\sum_{a=i,Z,e}\frac{n_{0a}e_a^2}{T_a}\frac{F_P' v_{\mathrm{th}a}}{V'}\int \mathrm{d}\lambda \oint \mathrm{d}\alpha \sum_{n=t_1,t_2,\ldots}\left(\frac{\hat{\tau}_{b\mathrm{th}}}{2}\overline{\omega}_{r\mathrm{th}}^2\right)_{n,a}}_{trapped\ particles}$$

MAGNETIC DIFFERENTIAL EQUATION

Magnetic differential equation (passing particles)

The following differential equation has to be solved:

$$v_{\|}\nabla_{\|}G = \omega_r - \overline{\omega}_r \ , \quad \omega_r = \vec{v}_d \cdot \nabla s$$

For passing particles $\overline{\omega}_r = 0$ so that $v_{\|}\nabla_{\|}G = \omega_r$.

In Boozer coordinates, this equation can be rewritten as follows:

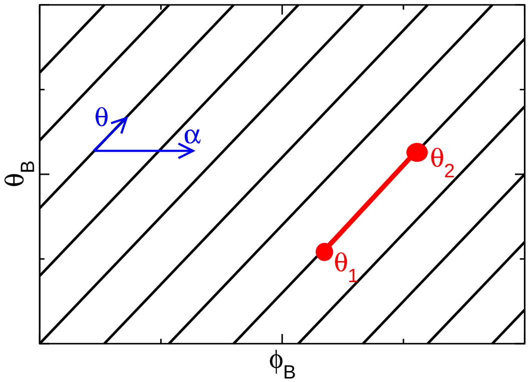
$$\left(F_P'\partial_\theta + F_T'\partial_\varphi\right) G = \left(I\partial_\theta - J\partial_\varphi\right)\rho_{\|} \ .$$

Finally, one finds the solution:

$$G|_{passing} = \underbrace{\frac{I}{F_P'}\rho_{\|}}_{tokamak} - \underbrace{\sum_{m,n\neq 0} \frac{qn}{m+qn}\left(1+\frac{J}{qI}\right)\frac{I}{F_P'}\rho_{\|mn}\exp(im\theta + in\varphi)}_{non-axisymmetric\ contribution} \ .$$

Magnetic differential equation (trapped particles)

$$\omega_r = \vec{v}_d \cdot \nabla s = \tau_b^{-1} \left[\frac{I}{F_P'} \left(\frac{\partial \rho_\parallel}{\partial \theta} \right)_\alpha - \underbrace{\frac{B^2\sqrt{g}}{F_P'^2} \left(\frac{\partial \rho_\parallel}{\partial \alpha} \right)_\theta}_{non-axisymmetric\ part} \right] = \omega_{r\theta} - \omega_{r\alpha} \,.$$



$\alpha = \varphi - q(s)\theta$ **is the field line label;**
θ **is taken along a fixed field line;**

$$\tau_b = (B\sqrt{g})/(F_P' v_\parallel)$$

$$\tau_b^{-1} \frac{\partial}{\partial \theta} \left(\frac{I}{F_P'} \rho_\parallel - G \right)_{\alpha=\text{const}} = \underbrace{\omega_{r\alpha} - \overline{\omega}_{r\alpha}}_{non-axisymmetric\ part} \,, \quad \overline{\omega}_{r\alpha} = \overline{\omega}_r$$

Magnetic differential equation (trapped particles)

Trapped-particle motion is periodic. Introduce a "time" variable:

$$\tau = \begin{cases} \sigma(\tau) \int\limits_{\theta_1}^{\theta} \tau_b \, \mathrm{d}\theta & : \quad 0 \le \tau \le \hat{\tau}_b/2 \\ \hat{\tau}_b/2 + \sigma(\tau) \int\limits_{\theta_2}^{\theta} \tau_b \, \mathrm{d}\theta & : \quad \hat{\tau}_b/2 \le \tau \le \hat{\tau}_b \end{cases}$$

Here, $\hat{\tau}_b$ is the bounce time. Introduce the bounce frequency $\hat{\omega}_b = 2\pi/\hat{\tau}_b$ and the "phase" of the trapped-particle oscillation $\zeta = \hat{\omega}_b \tau$.

Then, the magnetic differential equation for trapped particles takes the form:

$$\underbrace{\left(\hat{\rho} - \hat{G}\right) \frac{\partial \sigma}{\partial \zeta}}_{must\ vanish\ at\ \zeta = \pi n} + \sigma \frac{\partial}{\partial \zeta}\left(\hat{\rho} - \hat{G}\right) = \sigma \, \widetilde{\omega}_{r\alpha}/\hat{\omega}_b \, .$$

Here, we used notations $I\rho_{\|}/F_P' = \sigma\hat{\rho}$, $G = \sigma\hat{G}$.

Magnetic differential equation (trapped particles)

Trapped-particle motion is periodic in ζ with the period 2π.
A Fourier transform in ζ can be applied (bounce-harmonic expansion):

$$f(\zeta) = \sum_{l=-\infty}^{+\infty} f_l \exp(-il\zeta) \ , \quad f_l = \frac{1}{2\pi} \int_0^{2\pi} f(\zeta) \exp(il\zeta) \, \mathrm{d}\zeta \, .$$

Use the following "regularity condition":

$$\hat{G}(\pi n) = \hat{\rho}(\pi n) \quad \Longleftrightarrow \quad \hat{G}_0 = \hat{\rho}_0$$

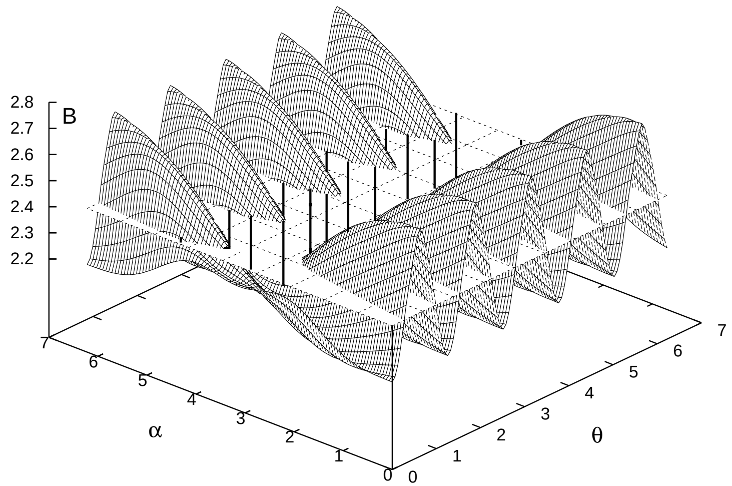
Solve the magnetic differential equation:

$$\frac{\partial}{\partial \zeta}\left(\hat{\rho} - \hat{G}\right) = \widetilde{\omega}_{r\alpha}/\hat{\omega}_b \quad \Longleftrightarrow \quad -il\left(\hat{\rho}_l - \hat{G}_l\right) = \omega_{rl}/\hat{\omega}_b \ , \quad l \neq 0 \, .$$

Magnetic differential equation (trapped particles)

Solution for the trapped particles:

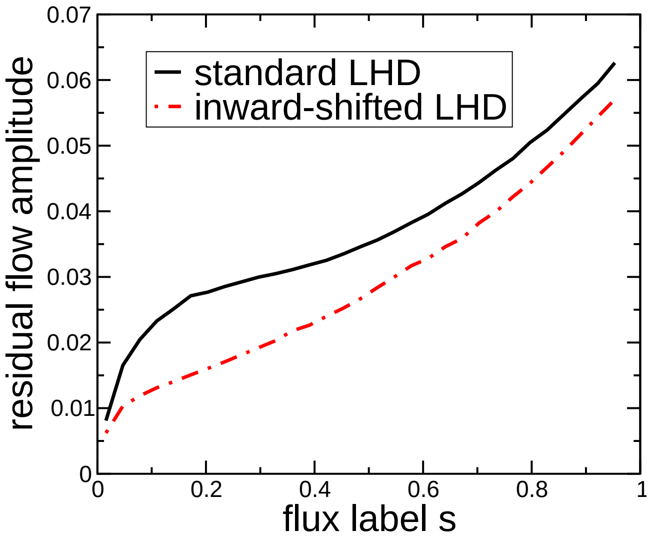
$$\overline{G^2} = \underbrace{\hat{\rho}_0^2 + 2\sum_{l>0} \hat{\rho}_l^2}_{tokamak\ term} + \underbrace{2\sum_{l>0} \omega_{rl}^2/(l^2\hat{\omega}_b^2)}_{non-axisymmetric\ contribution}$$

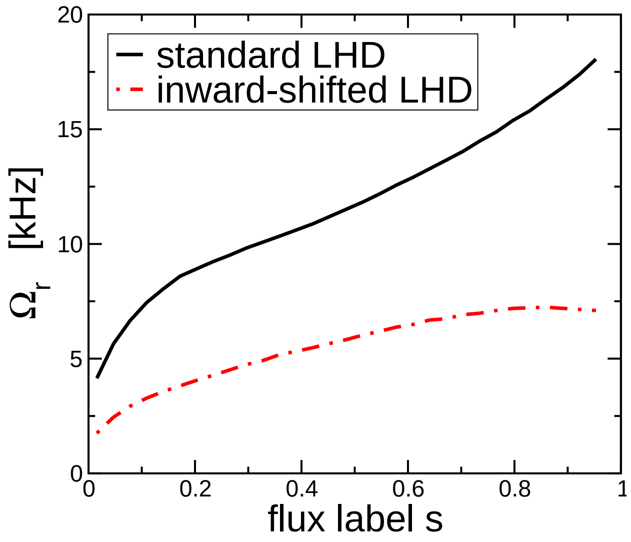


Compute the bounce harmonics numerically;
Take the numerical equilibrium (VMEC);
Solve numerically (using CAS3D-K)
the equation $1 - \lambda B(s, \theta_i, \alpha) = 0$;

for given pitch angle λ
on a given flux surface s

Zonal flows in stellarator geometry





Inward-shifted configuration is drift-optimized

Zonal flow frequency is reduced in the drift-optimized configuration

$$\Omega_r \sim \sqrt{\int \mathrm{d}\lambda \oint \mathrm{d}\alpha \;\; \hat{\tau}_b \, \overline{\omega}_r^2} \, , \quad \omega_r = \vec{v}_d \cdot \nabla s$$

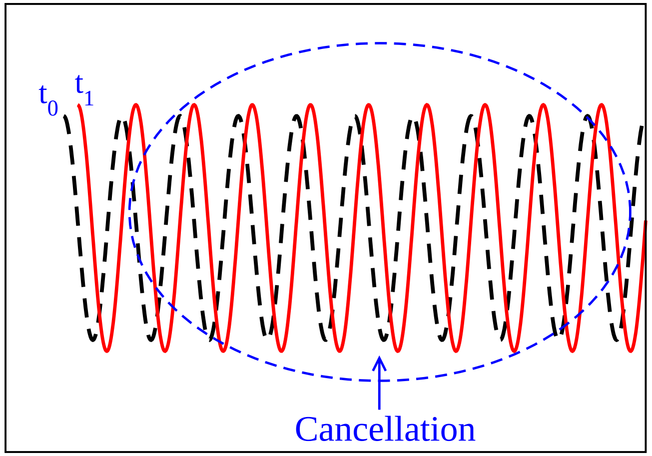
Zonal flows in stellarator geometry

Estimate response of our system to a noise source (Hinton-Rosenbluth)

$$\underbrace{\langle |\langle \phi_k(t)|^2\rangle\rangle}_{mean\ square\ potential} = \int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2 \ \underbrace{\langle\langle R_k(t_1) R_k(t_2)\rangle\rangle}_{noise\ corellation\ function} \ \underbrace{K_k(t_1)\, K_k(t_2)}_{linear\ response}$$

$\langle\langle R_k(t_1) R_k(t_2)\rangle\rangle \sim \exp\left(-\tau^2/\tau_c^2\right)$, $\tau = t_1 - t_2$, τ_c correlation time.

Linear response: $K_k(t) = \phi_k(t)/\phi_k(t=0) = A_R \cos\Omega_r t$



for $t \gg (1/\Omega, \tau_c)$ one can find

$$\langle\langle|\phi_k(t)|^2\rangle\rangle \sim A_R^2 \ \exp\left(-\frac{\Omega_r^2\tau_c^2}{4}\right).$$

Collisionless damping can be enhanced.

WHAT IS LEFT BEYOND

Higher-order FOW effects

We include FOW effects to the first order (only in continuity equation).

$$\frac{\partial \langle n_a \rangle}{\partial t} + \underbrace{\left\langle \int \mathrm{d}^3 v \, \vec{v}_{\mathrm{da}} \cdot \nabla f_{a1} \right\rangle}_{FOW \;\; effects} = 0$$

Distribution function is found neglecting FOW effects:

$$\frac{\partial f_{a1}}{\partial t} + v_{\|} \nabla_{\|} f_{a1} + \cancel{\vec{v}_d \cdot \nabla f_{a1}} = - \frac{e_a \phi'}{T_a} f_{a0} \left(\vec{v}_d \cdot \nabla s \right)$$

At the times $t \gg \tau_r = 1/(k_r v_d)$, higher-order FOW effects lead to a strong damping (through phase mixing in velocity space)

Important at sufficiently large $k_r v_d$ (short radial wavelengths)

H. Sugama and T.-H. Watanabe, Phys. Plasmas 13, 012501 (2006)

Effects due to background electric field

- Stellarators usually posses a background radial electric field $\vec{E}_0$ (neoclassics)

$$\frac{\partial f_{a1}}{\partial t} + v_{\|}\nabla_{\|} f_{a1} + \vec{v}_d \cdot \nabla f_{a1} + \vec{v}_E \cdot \nabla f_{a1} = -\frac{e_a \phi'}{T_a} f_{a0} \left(\vec{v}_d \cdot \nabla s\right)$$

- Trapped-particle orbits are modified by $\vec{E}_0$, forming e.g. "super-bananas"

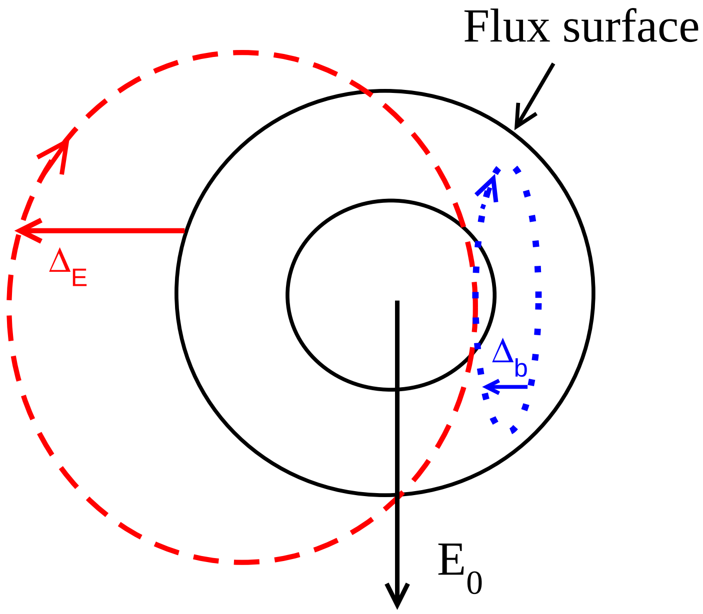
- Finite super-banana orbit width $\Delta_E \Longrightarrow$ additional ZF shielding

- Effect of the collisionless trapping-detrapping

H. E. Mynick and A. H. Boozer, Phys. Plasmas 14, 072507 (2007)

H. Sugama et al, Plasma and Fusion Research, 3, 041 (2008)

Transitioning orbits, collisions, electrons



trapped - passing orbit transitions
banana - superbanana orbit transitions

ZF shielding due to both Δ_b and Δ_E
(H. Sugama, private communication)

Collisionless detrapping
Collisional detrapping
Dependency of $\vec{E}_0$ on collisionality ν

Collisional damping of the residual zonal flow

Locally-trapped electrons (radial drifts are similar to that of ions)

Electron collisions may be relevant (in addition to ion collisions as in tokamaks)

Conclusions

- **Residual zonal flow dynamics is complicated in non-axisymmetric systems**
- **No steady-state solution after several bounce times ("Rosenbluth-Hinton equilibrium" in a tokamak)**
- **No time-scale separation between collisionless and collisional dynamics**