Effects of finite Larmor radius on equilibria with flow in reduced two-fluid models

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Outline of the talk

- Motivation
- Reduced fluid models for tokamak equilibria with flow
- Equilibrium with flow in reduced FLR two-fluid model - poloidal-Alfvenic flow [1]
- Single-fluid MHD equilibria with poloidal-sonic flow [1]
- Analytic single-fluid MHD equilibria with poloidal-sonic flow [2]
- Two-fluid FLR equilibria with poloidal-sonic flow
- Summary

[1] A. Ito, J.J. Ramos and N. Nakajima, Plasma Fusion Res. 3, 034 (2008)

[2] A. Ito and N. Nakajima, submitted to Plasma Phys. Control. Fusion

Motivation (1)

- Equilibrium with flow

  In improved confinement modes of magnetically confined plasmas where high-beta is achieved, equilibrium flows play important roles like the suppression of instability and turbulent transport .

- Characteristics of equilibria with flow are drastically altered by small scale effects.

- Equilibrium models with small scale effects may be suitable for initial states of multiscale simulation and for modeling steady states of improved confinement modes that have steep plasma profiles.

Motivation (2)

- Small-scale effects arising due to the Hall current and fluid closure problem have been studied with Hall MHD model.
  [A. Ito *et al.*, Phys. Plasmas **14**, 062502 (2007)]

  However, ions must be cold due to consistency with kinetic theory. Otherwise, the ion gyroviscosity and other finite ion Larmor radius (FLR) effects should be taken into account.

$\rho_i = d_i\sqrt{\beta_i}$, ρ_i : ion Larmor radius, d_i : ion skin depth, $\beta_i \equiv p_i \Big/ \left(B_0^2 / \mu_0 \right)$

- These hot ion effects may be relevant for high-temperature plasmas as in magnetic confinment fusion devices.
- Reduced fluid models

  By including higher-order terms of asymptotic expansions to reduced fluid models, the following effects on high-beta equilibria with flow can be easily studied:

  - Compressibility
  - Two-fluid effects
  - Ion finite Larmor radius effects

Equilibrium equations in FLR two-fluid model

- FLR two-fluid quations of steady state

$$\nabla\cdot(n\boldsymbol{v})=0,\quad \nabla\times\boldsymbol{E}=0,\quad \mu_0\boldsymbol{j}=\nabla\times\boldsymbol{B}$$

$$\boldsymbol{m_i n v}\cdot\nabla\boldsymbol{v}=\boldsymbol{j}\times\boldsymbol{B}-\nabla(p_i+p_e)-\lambda_i\nabla\cdot\Pi_i^{gv},$$

$$\boldsymbol{E}=-\boldsymbol{v}\times\boldsymbol{B}+\left(\lambda_H/ne\right)[\boldsymbol{j}\times\boldsymbol{B}-\nabla p_e],$$

$$\boldsymbol{v}\cdot\nabla p_i+\gamma p_i\nabla\cdot\boldsymbol{v}+\left(2/5\right)\lambda_i\gamma\nabla\cdot\boldsymbol{q}_i=0,$$

$$\left(\boldsymbol{v}-\boldsymbol{j}/ne\right)\cdot\nabla p_e+\gamma p_e\nabla\cdot\left(\boldsymbol{v}-\boldsymbol{j}/ne\right)+\left(2/5\right)\lambda_e\gamma\nabla\cdot\boldsymbol{q}_e=0,$$

$$\boxed{\begin{aligned}(\lambda_i,\lambda_e,\lambda_H)&=(0,0,0)\quad\Rightarrow\quad\text{Single - fluid MHD}\\ &=(0,0,1)\quad\Rightarrow\quad\text{Hall MHD}\\ &=(1,1,1)\quad\Rightarrow\quad\text{FLR two - fluid MHD}\end{aligned}}$$

- Hot ion effects: gyroviscosity Π_i^{gv}, diamagnetic heat flux $\boldsymbol{q}_{i\perp}$

  - Obtained from Vlasov equation by asymptotic expansions in terms of the small parameter $\delta\sim\rho_i/a\ll1$

  - Much simplified in the slow dynamics ordering $\delta\sim v/v_{th}$

- Reduced equations for FLR two-fluid equilibria

  Asymptotic expansions of equilibrium equations in terms of inverse aspect ratio $\varepsilon\equiv a/R\ll1$ are suitable for including hot ion effects in asymptotic expansions

Assumptions for reduced equations

- Compressible high-β tokamak and slow dynamics orderings
  - Large aspect ratio and high-β

$$\varepsilon \equiv a/R \ll 1, \quad B_p \sim \varepsilon B_0, \quad p \sim \varepsilon \left(B_0^2/\mu_0\right)$$

  - Compressibility $\nabla \cdot \boldsymbol{v}_{MHD} \sim \varepsilon v_{MHD}/a\,, \quad \left(\boldsymbol{v} \equiv \boldsymbol{v}_{MHD} + \boldsymbol{v}_{di}\right)$
    to eliminate the fast magnetosonic wave
  - Flow velocity for slow dynamics

$$v \sim v_{MHD} \sim v_{di} \sim \delta v_{th}, \quad \left|\nabla \cdot \Pi^{gv}\right| \sim \delta^2 \left|\nabla p\right| \sim \delta^2 m_i n v_{th}^2$$

- Characteristic flow velocities
  - Poloidal Alfven velocity $v^2 \sim \varepsilon v_{th}^2 \sim v_{Ap}^2 \quad \Rightarrow \boxed{\delta^2 \sim \varepsilon}$
    Alfven singularity at $v_p^2 = v_{Ap}^2$
    Standard reduced MHD ordering [1,2] applies
  - Poloidal sound velocity $v^2 \sim \left(B_p/B_0\right)^2 \gamma p/\rho \sim \varepsilon v_{Ap}^2 \quad \Rightarrow \boxed{\delta \sim \varepsilon}$
    Transition between sub- and super-poloidal-sonic flow
    Higher order terms should be taken into account

[1] Strauss, Phys. Fluids **20**, 1354 (1977)
[2] Hazeltine *et al.*, Phys. Fluids 28, 2466 (1985)

Equilibria with flow in reduced two-fluid models

- Axisymmetry $\partial/\partial\varphi = 0, \quad \boldsymbol{B} = \nabla\psi\times\nabla\varphi + I\nabla\varphi$
- Projections of the force balance on $\nabla\psi, \boldsymbol{B}, \boldsymbol{B}_p$

$$\mu_0 R^2\nabla\psi\cdot(m_i n\boldsymbol{v}\cdot\nabla\boldsymbol{v} + \lambda_i\nabla\cdot\Pi_i^{gv}) + |\nabla\psi|^2\ \Delta^*\psi + I\nabla\psi\cdot\nabla I + \mu_0 R^2\nabla\psi\cdot\nabla(p_i + p_e) = 0,$$

$$\boldsymbol{B}\cdot(m_i n\boldsymbol{v}\cdot\nabla\boldsymbol{v} + \lambda_i\nabla\cdot\Pi_i^{gv}) + \{p_i + p_e, \psi\} = 0,$$

$$(\nabla\psi\times\nabla\varphi)\cdot(m_i n\boldsymbol{v}\cdot\nabla\boldsymbol{v} + \lambda_i\nabla\cdot\Pi_i^{gv}) + \{p_i + p_e, \psi\} + (I/\mu_0 R^2)\{I,\psi\} = 0,$$

$$\{a, b\} \equiv (\nabla a\times\nabla b)\cdot\nabla\varphi$$

- Asymptotic expansions

$$f = f_0 + f_1 + f_2 + f_3 + \cdots,\ f_1 \sim \varepsilon f_0,\ f_2 \sim \varepsilon^2 f_0,\ f_3 \sim \varepsilon^3 f_0$$

$$\psi = \psi_1 + \psi_2 + \psi_3 + \ldots, \quad I = I_0 + I_1 + I_2 + I_3 + \ldots,$$

$$p_i = p_{i1} + p_{i2} + p_{i3} + \ldots, \quad p_e = p_{e1} + p_{e2} + p_{e3} + \ldots,$$

$$n = n_0 + n_1 + \ldots, \quad R = R_0 + x,$$

- Leading order of the force balance

$$\frac{B_0}{\mu_0 R_0} I_1 + p_{i1} + p_{e1} = const.$$

Equilibrium with flow in reduced FLR two-fluid model - poloidal-Alfvenic flow

[A. Ito *et al.*, Plasma Fusion Res. **3**, 034 (2008)]

- Second order accuracy for the total energy is required

$$m_i n v^2 \sim \left\|\Pi_i^{gv}\right\| \sim \varepsilon p \sim \rho v_{Ap}^2 \sim \varepsilon^2 \left(B_0^2/\mu_0\right), \quad v \sim j/ne \sim \nabla p/neB_0$$

- Ion flow velocity

obtained from the generalized Ohm's law:

$$\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B} = \frac{\lambda_H}{ne}\left(\nabla p_i \underline{+ m_i n \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \lambda_i \nabla\cdot\Pi_i^{gv}}\right)$$

by taking the lowest order terms:

$$\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B} \simeq \frac{\lambda_H}{ne}\nabla p_i$$

$$\boldsymbol{v} \equiv \boldsymbol{v}_{\mathrm{MHD}} + \boldsymbol{v}_{di}$$

$$\boldsymbol{v}_{\mathrm{MHD}} \simeq -\frac{1}{B_0}\nabla\Phi_1 \times \left(R_0\nabla\varphi\right) + v_\parallel R_0 \nabla\varphi, \quad \boldsymbol{v}_{di} \simeq -\frac{1}{eB_0 n_0}\nabla p_{i1} \times \left(R_0\nabla\varphi\right),$$

$$\nabla\cdot\boldsymbol{v} \simeq -\left(\lambda_H R_0/eB_0\right)\{n_0^{-1}, p_{i1}\}, \qquad \{a, b\} \equiv (\nabla a \times \nabla b)\cdot\nabla\varphi$$

$\boldsymbol{v}_{\mathrm{MHD}}$: ExB and parallel flow $\boldsymbol{v}_{di}$: ion diamagnetic flow

Higher-order derivative term of the Hall current is neglected.

- Ion heat flux $\nabla\cdot\boldsymbol{q}_i \simeq \nabla\cdot\boldsymbol{q}_{i\perp} \simeq \frac{5}{2}\frac{p_{i1}R_0}{eB_0}\{n_0^{-1}, p_{i1}\}.$

- Ion gyroviscous force for collisionless magnetized plasma

  [Ramos, Phys. Plasmas **12**, 112301 (2005)]

$$\nabla\cdot\Pi_i^{gv} = \nabla\cdot\left(\sum_{N=1}^{5}\Pi_i^{gvN}\right)$$

$$\nabla\cdot\Pi_i^{gv1} \simeq -m_i n\mathbf{v}_{*i}\cdot\nabla\mathbf{v} - \nabla\chi_v + \frac{m_i R_0^2}{4e^2B_0^2}\nabla\varphi\times\nabla\left\{\nabla\varphi\cdot\left[\nabla p_i^2\times\nabla\left(\frac{1}{n}\right)\right]\right\},$$

$$\nabla\cdot\Pi_i^{gv2} \simeq -\nabla\chi_q + \frac{m_iR_0}{4eB_0}\nabla\varphi\times\nabla\left(\frac{4}{5}\nabla\cdot\mathbf{q}_{i\perp}\right)$$

$$\simeq -\nabla\chi_q - \frac{m_iR_0^2}{4e^2B_0^2}\nabla\varphi\times\nabla\left\{\nabla\varphi\cdot\left[\nabla p_i^2\times\nabla\left(\frac{1}{n}\right)\right]\right\},$$

$$\nabla\cdot\Pi_i^{gv3} \simeq \nabla\cdot\Pi_i^{gv4} \simeq \nabla\cdot\Pi_i^{gv5} \simeq 0,$$

$$\mathbf{v}_{*i} \simeq \frac{R_0}{enB_0}\nabla\varphi\times\nabla p_i,\quad \chi_v \equiv \frac{m_ip_i}{2eB^2}\mathbf{B}\cdot(\nabla\times\mathbf{v}),\quad \chi_q \equiv \frac{m_i}{5eB^2}\mathbf{B}\cdot(\nabla\times\mathbf{q}_{i\perp}).$$

$$\underline{\nabla\cdot\Pi_i^{gv} \simeq -\frac{m_i}{eB_0}\left(R_0\nabla\varphi\times\nabla p_{i1}\right)\cdot\nabla\boldsymbol{v} - \nabla\left(\chi_v + \chi_q\right)}$$

- Lowest order quantities are arbitrary functions of ψ_1

$$\Phi_1 = \Phi_1(\psi_1),\quad n_0 = n_0(\psi_1),\quad p_{i1} = p_{i1}(\psi_1),\quad p_{e1} = p_{e1}(\psi_1),\quad I_1 = I_1(\psi_1).$$

- Poloidal force balance

$$p_{i2} + p_{e2} + \frac{B_0}{\mu_0R_0}I_2 + \frac{m_in_0}{B_0^2}\left[\Phi_1' + \frac{(\lambda_e-\lambda_i)p_{i1}'}{en_0}\right]\left(\Phi_1' + \frac{\lambda_ep_{i1}'}{en_0}\right)\frac{|\nabla\psi_1|^2}{2} - \lambda_i(\chi_v+\chi_q) \equiv g_*(\psi_1).$$

- Reduced Grad-Shafranov equation with flow and FLR

$$\left[1-M_{Ap}\left(M_{Ap}-\lambda_i\frac{v_{di}}{v_{Ap}}\right)\right]\Delta_2\psi_1-\frac{\left|\nabla\psi_1\right|^2}{2}\left[M_{Ap}\left(M_{Ap}-\lambda_i\frac{v_{di}}{v_{Ap}}\right)\right]'=-\mu_0R_0^2\left[\frac{2x}{R_0}\left(p_{i1}+p_{e1}\right)'+g_*'\right]-\left(\frac{I_1^2}{2}\right)'$$

$$M_{Ap}\left(\psi_1\right)\equiv\sqrt{\mu_0m_in_0}R_0\left(\frac{\Phi_1'}{B_0}-\frac{\lambda_Hp_{i1}'}{en_0B_0}\right):$$ Poloidal Alfven Mach number

$$\left(V_{di}/V_{Ap}\right)\left(\psi_1\right)\equiv\sqrt{\mu_0m_in_0}R_0\left(p_{i1}'/en_0B_0\right),\qquad\Delta_2\equiv\left(\frac{\partial^2}{\partial R^2}+\frac{\partial^2}{\partial Z^2}\right)$$

- Alfven singularity

-Single-fluid and Hall MHD: $M_{Ap}^2=1$

-FLR two-fluid: $M_{Ap}=\frac{1}{2}\left[v_{di}/v_{Ap}\pm\sqrt{4+\left(v_{di}/v_{Ap}\right)^2}\right]$

Singularity is shifted by the gyroviscous cancellation

- Comparison with non-reduced Hall MHD equilibrium

-Resolution of the singularity by the Hall current does not occur in the leading order because the higher-order derivative term is neglected.

-It may be predicted that the singular point shifted by FLR effect is resolved by singular perturbation due to Hall current.

Single-fluid equilibria with poloidal-sonic flow

[A. Ito *et al.*, Plasma Fusion Res. **3**, 034 (2008)]

- Third order accuracy for the total energy is required

$$m_i n v^2 \sim \left(B_p/B_0\right)^2 \gamma p \sim \varepsilon^3 \left(B_0^2/\mu_0\right)$$

- Lowest order quantities are arbitrary functions of ψ_1

$$n_0 = n_0(\psi_1), \quad p_1 = p_1(\psi_1), \quad I_1 = I_1(\psi_1).$$

- Flow velocity $\boldsymbol{v} \equiv \left(1 + x/R_0\right) \nabla U \times \left(\boldsymbol{B}/B\right) + v_\parallel \left(\boldsymbol{B}/B\right)$
  to satisfy both $\nabla \cdot \boldsymbol{v} \sim \varepsilon v/a$ and, from Faraday's law,

$$\boldsymbol{v} \cdot \nabla p_1 \simeq \left(B_0^2/\mu_0\right) R^2 \nabla \cdot \left(\boldsymbol{v}_\perp/R^2\right) \sim \varepsilon^2 \left(B_0^2/\mu_0\right)\left(v/a\right)$$

[Strauss, Nucl. Fusion **23**, 649 (1983)]

Asymptotic expansion

$$U = U_1 + U_2 + \cdots, \quad (\nabla \cdot \boldsymbol{v}_1) = \left\{\left(v_\parallel/B_0\right) + 2xU_1', \psi_1\right\}.$$

$$U_1 = U_1(\psi_1), \quad U_2 = U_1'(\psi_1)\psi_2 + U_{2*}(\psi_1) \qquad (\ )' \equiv d(\ )/d\psi_1$$

- Poloidal force balance

$$p_2 + \frac{B_0}{\mu_0 R_0} I_2 \equiv g_*(\psi_1),$$

$$p_3 + \frac{B_0 I_3}{\mu_0 R_0} + \frac{I_1}{\mu_0 R_0^2}(I_2 - I_1' \psi_2) + m_i n_0 U_1'^2 \frac{|\nabla \psi_1|^2}{2} + \left(\frac{x}{R_0}\right)^2 \frac{2M_{Ap}^2 \gamma p_1}{\beta_1 - M_{Ap}^2} - g_*' \psi_2 \equiv E_*(\psi_1).$$

- Pressure equation $\boldsymbol{v} \cdot \nabla p = -\gamma p \nabla \cdot \boldsymbol{v}$

$$p_1 = p_1(\psi_1)$$

$$p_2 - p_1{}'(\psi_1)\psi_2 + \gamma p_1 \left[\frac{v_\parallel}{B_0 R_0 U_1{}'(\psi_1)} + \frac{2x}{R_0}\right] \equiv p_{2*}(\psi_1)$$

- Equation for continuity $\nabla \cdot (\rho \boldsymbol{v}) = 0$

$$n_0 = n_0(\psi_1)$$

$$n_1 - n_0{}'(\psi_1)\psi_2 + \frac{n_0 v_\parallel}{B_0 R_0 U_1{}'(\psi_1)} + \frac{2x}{R_0} n_0 \equiv n_*(\psi_1)$$

- Force balance in $\boldsymbol{B}$

$$B_0 R_0 m_i n_0 U_1{}'(\psi_1) v_\parallel + p_2 - p_1{}'(\psi_1)\psi_2 \equiv p_{3*}(\psi_1)$$

Coupled by slow magnetosonic wave

Asymptotic equations for single-fluid equilibria with poloidal-sonic flow

- Magnetic flux $\psi \simeq \psi_1 + \psi_2, \quad \psi_2 \sim \varepsilon\psi_1$

Asymptotic expansion of the generalized Grad-Shafranov equation

$$\Delta_2\psi_1 = -\mu_0 R_0^2\left[\left(2x/R_0\right)p_1{}' + g_*{}'\right] - \left(I_1^2/2\right)',$$

$$\Delta_2\psi_2 + \left[\mu_0 R_0^2\left(\frac{2x}{R_0}p_1'' + g_*''\right) + \left(\frac{I_1^2}{2}\right)''\right]\psi_2$$

$$= \frac{1}{R}\frac{\partial\psi_1}{\partial R} + M_{Ap}^2(\psi_1)\Delta_2\psi_1 + \frac{\left|\nabla\psi_1\right|^2}{2}\left(M_{Ap}^2\right)' - \mu_0 R_0^2\left[\left(\frac{x}{R_0}\right)^2 p_1{}'(\psi_1) + E_*{}'(\psi_1)\right]$$

$$-\mu_0 R_0^2\left[\left(\frac{x}{R_0}\right)^2\left(\frac{2M_{Ap}^2\gamma p_1}{\beta_1 - M_{Ap}^2}\right)' - \frac{2x}{R_0}\left(\frac{M_{Ap}^2 p_{2*} - \beta_1 p_{2*}}{\beta_1 - M_{Ap}^2}\right)'\right]$$

$$\Delta_2 \equiv \left(\frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial Z^2}\right), \quad \beta_1(\psi_1) \equiv \frac{\gamma p_1(\psi_1)}{B_0^2/\mu_0},$$

$$M_{Ap}(\psi_1) \equiv \sqrt{\mu_0\rho_0(\psi_1)}R_0U_1'(\psi_1) : \text{poloidal Alfven Mach number}$$

- Parallel flow $v_\parallel$: $v_\parallel = -\left(\frac{2x}{R_0}\right)\frac{\beta_1 M_{Ap} v_A}{\beta_1 - M_{Ap}^2}$,

- Pressure $p \simeq p_1 + p_2$: $p_1 = p_1(\psi_1)$,

$$p_2 = p_1'(\psi_1)\psi_2 + \left(\frac{2x}{R_0}\right)\frac{M_{Ap}^2 \gamma p_1}{\beta_1 - M_{Ap}^2} - \frac{M_{Ap}^2 p_{2*} - \beta_1 p_{3*}}{\beta_1 - M_{Ap}^2}$$

- Density $n \simeq n_0 + n_1$: $n_0 = n_0(\psi_1)$,

$$n_1 = n_0'\psi_2 + n_{1*} + \left(\frac{2x}{R_0}\right)\frac{M_{Ap}^2}{\beta_1 - M_{Ap}^2}n_0 - \frac{p_{2*} - p_{3*}}{(\beta_1 - M_{Ap}^2)(B_0^2/\mu_0)}n_0,$$

- Poloidal-sonic singularity $\beta_1 - M_{Ap}^2$

-Singularity appears when the poloidal flow velocity equals poloidal sound velocity

-Higher-order equations in ε must be taken into account to include this singularity in the reduced model

-Shock structure does not appear because of the ordering $\left(\beta_1 - M_{Ap}^2\right)/\beta_1 \sim 1$

cf. shock ordering $\left(\beta_1 - M_{Ap}^2\right)/\beta_1 \leq \sqrt{\varepsilon}$

[Shaing *et al*. Phys. Fluids B **4**, 404 (1992)]

Analytic solutions of asymptotic equations for single-fluid equilibrium with poloidal-sonic flow

[A. Ito and N. Nakajima, submitted to Plasma Phys. Control. Fusion]

- Reduced GS equations for MHD equilibria with poloidal-sonic flow can be solved analytically for linear profiles
- The solutions include both of the following effects

  - poloidal flow comparable to the poloidal sound velocity
  - high-β
- Other analytic equilibria with flow

  - purely toroidal flow [1]
  - incompressible flow [2]
  - low-β, poloidal-sonic flow [3,4]
  - high-β, poloidal-Alfvenic flow [5]

[1] Maschke and Perrin, Plasma Phys. **22**, 579 (1980)
[2] Tasso and Throumoulopoulos, Phys. Plasmas **5**, 2378 (1998)
[3] Shaing *et al*. Phys. Fluids B **4**, 404 (1992)
[4] Betti and Freidberg, Phys. Plasmas **7**, 2439 (2000)
[5] Guazzotto and Betti, Phys. Plasmas **12** 056107 (2005)

- Linear profiles:

$$p_1 = p_{1c}\left(\psi_1 / \psi_c\right), \quad g_* + \frac{I_1^2}{2\mu_0 R_0^2} = g_c\left(\psi_1 / \psi_c\right),$$

$$M_{Ap}^2 \equiv \mu_0 m_i n_0 \left(R_0 U_1'\right)^2 = M_{Apc}^2\left(\psi_1 / \psi_c\right),$$

$E_* = p_{2*} = p_{3*} = 0, \quad \psi_c,\, p_{1c},\, g_c,\, M_{Apc}$ and β_c are constant

  - No poloidal sonic singularity: $\gamma p_{1c} \big/ \left(B_0^2/\mu_0\right) \neq M_{Apc}^2$
  - n_0 and U_1' remain arbitrary as long as M_{Ap}^2 satisfies the above relation

- Normalized equations in toroidal coordinates (r, θ, ϕ)

$$\Delta_2 \psi_1 = -\frac{g_c}{B_p^2} - 2\frac{p_{1c}}{B_p^2} r\cos\theta,$$

$$\Delta_2 \psi_2 = \left[\cos\theta\left(\partial\psi_1 / \partial r\right) - \left(\sin\theta / r\right)\left(\partial\psi_1 / \partial\theta\right)\right] + M_{Apc}^2\left(\psi_1 \Delta_2 \psi_1 + \left|\nabla\psi_1\right|^2 / 2\right)$$

$$+C\left[1 + 2\gamma M_{Apc}^2 \big/ \left(\gamma p_{1c} - M_{Apc}^2\right)\right] r^2 \cos^2\theta$$

$$r \rightarrow \overline{r},\; \psi_1 \rightarrow \psi_c \overline{\psi_1},\; \psi_2 \rightarrow \varepsilon\psi_c \overline{\psi_2},\; a \rightarrow \varepsilon R_0,\; V_{Ec}^2 \rightarrow \varepsilon \overline{V}_{Ec}^2$$

$$\psi_c / B_0 R_0 a \rightarrow \varepsilon B_p,\; p_{1c} \rightarrow \varepsilon\left(B_0^2 / \mu_0\right) p_{1c},\; g_c \rightarrow \varepsilon^2\left(B_0^2 / \mu_0\right) g_c,\; p_{1c} \rightarrow \varepsilon p_{1c}$$

- Boundary conditions: $\quad \psi_1(1) = 0, \quad \psi_2(1) = 0.$

- Analytic solutions for $\psi \simeq \psi_1 + \psi_2$

$$\psi_1 = -\frac{1}{4}\left(r^2 - 1\right)\left[\left(\frac{g_c}{B_p^2}\right) + \left(\frac{p_{1c}}{B_p^2}\right) r\cos\theta\right].$$

- Identical to static equilibria derived by Haas, Phys. Fluids **15**, 141 (1972)

$$\psi_2 = P(r) + Q(r)\cos\theta + R(r)\cos 2\theta,$$

$$P(r) = \frac{1}{16}(r - 1^2)\left\{-\left(\frac{p_{1c}}{B_p^2}\right)\left[r^2 + \frac{\gamma M_{Apc}^2}{\gamma p_{1c} - M_{Apc}^2}(r^2 + 1)\right] + \frac{M_{Apc}^2}{8}\left[\frac{1}{9}\left(\frac{p_{1c}}{B_p^2}\right)^2\left(13r^4 - 14r^2 - 5\right) + \left(\frac{g_c}{B_p^2}\right)^2(3r^2 - 5)\right]\right\}$$

$$Q(r) = -\frac{1}{16}\frac{g_c}{B_p^2} r(r^2 - 1)\left[1 - \frac{M_{Apc}^2}{4}\frac{p_{1c}}{B_p^2}(3r^2 - 4)\right]$$

$$R(r) = -\frac{1}{16}\frac{p_{1c}}{B_p^2} r^2(r^2 - 1)\left[1 + \frac{4}{3}\frac{\gamma M_{Apc}^2}{\gamma p_{1c} - M_{Apc}^2} - \frac{M_{Apc}^2}{48}\frac{p_{1c}}{B_p^2}(9r^2 - 11)\right]$$

- Generalization of the static equilibria derived by Yamazaki *et al.*, Jpn. J. Appl. Phys. **18**, 981 (1979) to include poloidal-sonic flows
- Solutions for the vacuum region has been derived as well

- Poloidal-beta value $\nu \equiv p_{1c}/g_c$

$$\beta_p \equiv \frac{8\pi^2 a^2}{\mu_0 I_\varphi^2}\langle p\rangle \simeq \frac{\nu}{\varepsilon}\left\{1+\varepsilon\left[\frac{7\nu}{12}-\frac{\nu}{3}\frac{\gamma M_{Apc}^2}{\gamma p_{1c}-M_{Apc}^2}+\frac{M_{Apc}^2}{64}\frac{g_c}{B_p^2}\left(6-\nu^2\right)\right]\right\}$$

- Shafranov shift $\overline{\Delta}_s \simeq \overline{\Delta}_{s1} + \varepsilon\overline{\Delta}_{s2},$

$$\overline{\Delta}_{s1} = \frac{-1+\sqrt{1+3\nu^2}}{3\nu},$$

$$\overline{\Delta}_{s2} = -\frac{\partial\overline{\psi}_2}{\partial\overline{r}}\bigg|_{\overline{r}=\overline{\Delta}_{s1},\theta=0} \bigg/ \frac{\partial^2\overline{\psi}_1}{\partial\overline{r}^2}\bigg|_{\overline{r}=\overline{\Delta}_{s1},\theta=0} = \left(\frac{2B_p^2}{g_c}\right)\frac{P'\left(\overline{\Delta}_{s1}\right)+Q'\left(\overline{\Delta}_{s1}\right)+R'\left(\overline{\Delta}_{s1}\right)}{1+3\nu\overline{\Delta}_{s1}}.$$

- Shift of the pressure maximum $\overline{\Delta}_p \simeq \overline{\Delta}_s + \varepsilon\overline{\Delta}_{p2},$

$$\overline{\Delta}_{p2} = \frac{\gamma M_{Apc}^2}{\gamma p_{1c}-M_{Apc}^2}\frac{1+\nu\overline{\Delta}_{s1}}{1+3\nu\overline{\Delta}_{s1}}\left(1-\overline{\Delta}_{s1}^2\right),$$

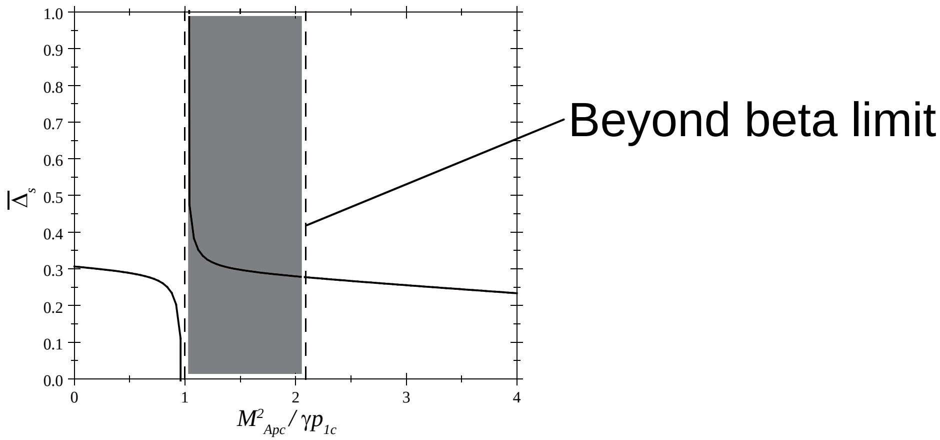
- Equilibrium beta limit

  The condition that the separatrix does not appear in the plasma region

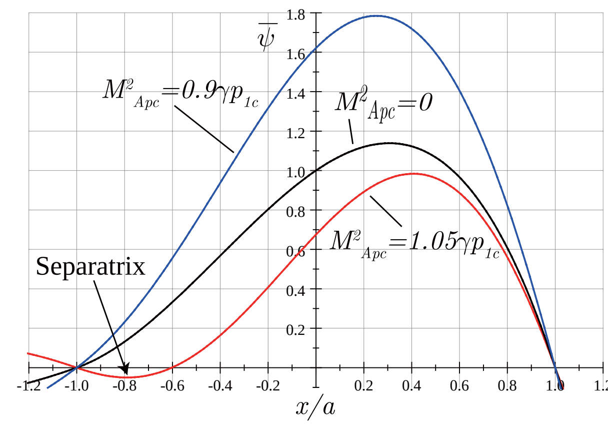
$$\varepsilon\beta_p < 1+\varepsilon\left(\frac{5}{6}+\frac{17M_{Apc}^2}{192}\frac{g_c}{B_p^2}-\frac{1}{6}\frac{\gamma M_{Apc}^2}{\gamma g_c-M_{Apc}^2}\right).$$

- Shafranov shift is modified to yield a forbidden region and the pressure surface departs from magnetic surface due to poloidal-sonic flow.

- Shafranov shift

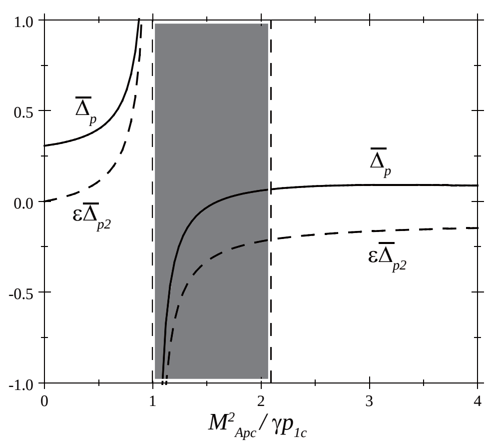


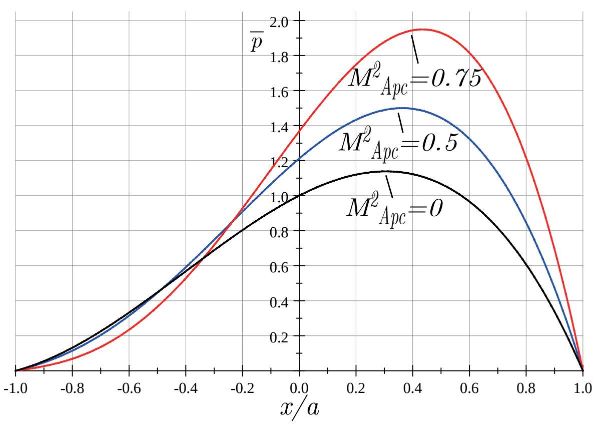
Profile of ψ at midplane

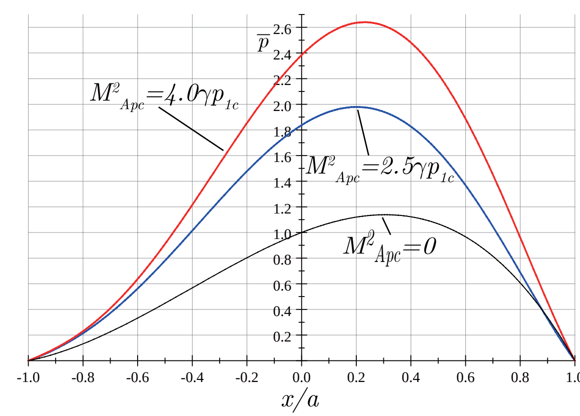


- Shift of the pressure maximum

Profile of p at midplane







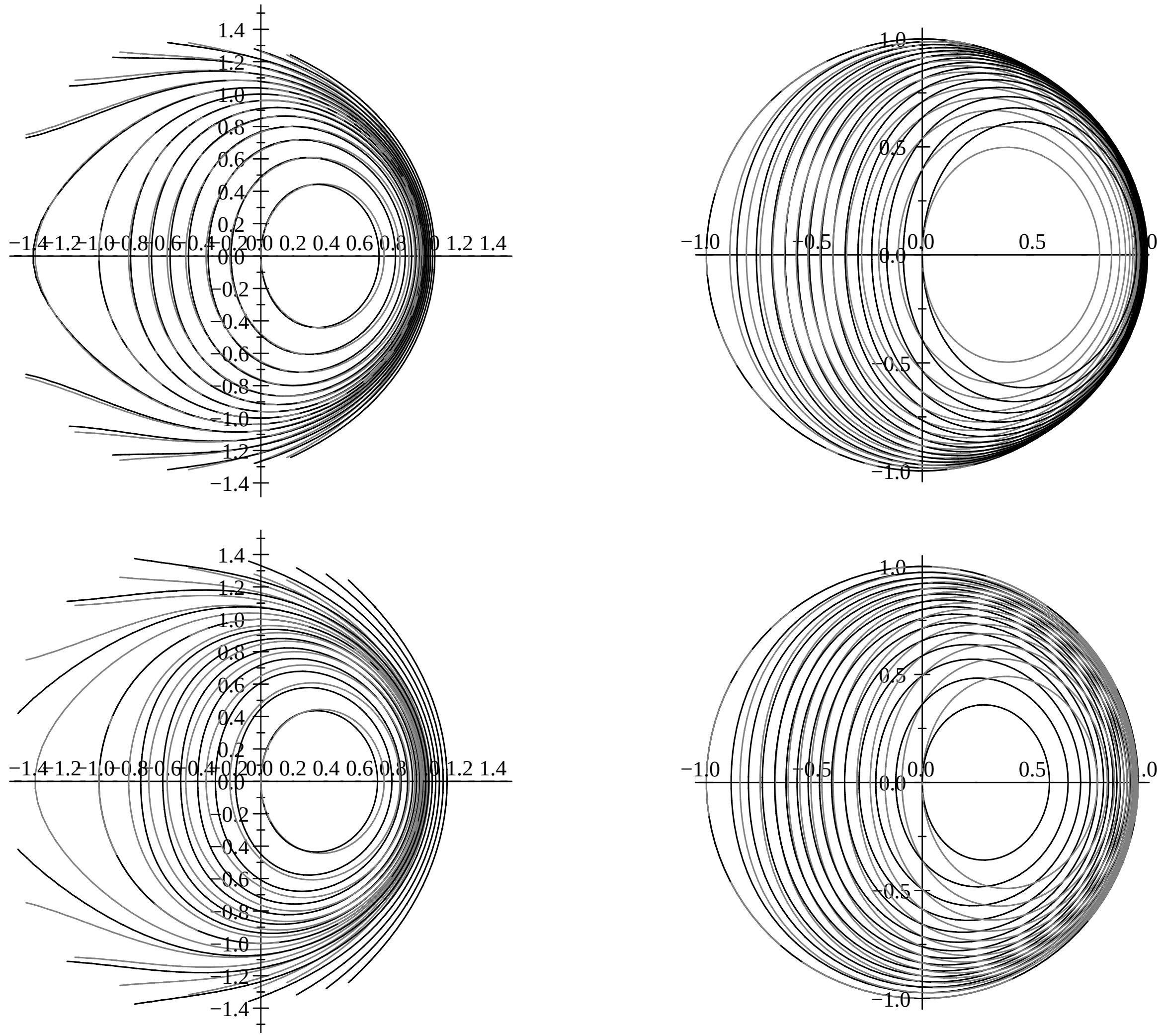
Magnetic surface (gray: static equilibrium) | Pressure surface (gray: magnetic surface)

Sub-poloidal -sonic flow

$\left(\gamma p_{1c} / M_{Apc}^2 = 0.5\right)$

Super-poloidal -sonic flow

$\left(\gamma p_{1c} / M_{Apc}^2 = 2.5\right)$



- The pressure maximum is shifted outwards for sub-poloidal-sonic flow and inwards for super-poloidal-sonic flow

Equilibrium with poloidal-sonic flow in reduced two-fluid models

- Orderings

  - Slow dynamics and poloidal-sonic flow

$$v^2 \sim \left(B_p / B_0\right)^2 \gamma p / \rho \sim \varepsilon v_{Ap}^2 \;\Rightarrow\; \boxed{\delta \sim \varepsilon}$$

- Lowest order quantities are arbitrary functions of ψ_1

$$\Phi_1 = \Phi_1(\psi_1), \quad n_0 = n_0(\psi_1), \quad p_{i1} = p_{i1}(\psi_1), \quad p_{e1} = p_{e1}(\psi_1), \quad I_1 = I_1(\psi_1).$$

- Ion flow velocity

  obtained from the generalized Ohm's law:

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = \frac{\lambda_H}{ne}\left(\nabla p_i \underline{+ m_i n \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \lambda_i \nabla \cdot \Pi_i^{gv}}\right)$$

up to the first order,

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} \simeq \frac{\lambda_H}{ne} \nabla p_i$$

Higher-order derivative term of the Hall current is neglected.

- Ion flow compressibility

$$(\nabla\cdot\mathbf{v})^{(0)}=0$$

$$(\nabla\cdot\mathbf{v})^{(1)}=\frac{1}{B_0}\left\{v_\parallel-\left(\frac{2x}{R_0}\right)R_0\left(\Phi_1'+\lambda_H\frac{p_{i1}'}{en_0}\right)+\lambda_H\frac{R_0p_{i1}'}{en_0^2}n_1-\lambda_H\frac{R_0n_0'}{en_0^2}p_{i2},\psi_1\right\}.$$

- Ion heat flux $(\nabla\cdot\boldsymbol{q}_i)^{(0)}=(\nabla\cdot\boldsymbol{q}_{i\perp})^{(0)}=\frac{5}{2}\frac{p_{i1}R_0}{eB_0}\{n_0^{-1},p_{i1}\},$

- Higher-order terms are needed

$$(\nabla\cdot\mathbf{q}_{i\perp})^{(1)}\simeq\frac{5R_0p_{i1}}{2en_0B_0}\left\{\frac{n_0'}{n_0}p_{i2}-\frac{p_{i1}'}{n_0}n_1-\left(\frac{2x}{R_0}\right)\left(p_{i1}'-\frac{p_{i1}n_0'}{n_0}\right),\psi_1\right\}.$$

- Parallel heat flux is neglected for simplicity

- Ion gyroviscosity

$$\nabla\cdot\Pi_i^{gv}\simeq-\frac{m_i}{eB_0}\left(R_0\nabla\varphi\times\nabla p_{i1}\right)\cdot\nabla\boldsymbol{v}-\nabla\left(\chi_v+\chi_q\right),$$

- Poloidal force balance

$$p_{i2} + p_{e2} + \frac{B_0}{\mu_0 R_0} I_2 \equiv g_* (\psi_1),$$

$$p_{i3} + p_{e3} + \frac{B_0 I_3}{\mu_0 R_0} + \frac{I_1}{\mu_0 R_0^2}\left(I_2 - I_1^{'}\psi_2\right) + \frac{m_i n_0}{B_0^2}\left[\Phi_1^{'} + \frac{(\lambda_H - \lambda_i)p_{i1}^{'}}{en_0}\right]\left(\Phi_1^{'} + \frac{\lambda_H p_{i1}^{'}}{en_0}\right)\frac{|\nabla\psi_1|^2}{2}$$

$$-\lambda_i(\chi_v + \chi_q) + \frac{1}{2}\left(\frac{2x}{R_0}\right)^2 \frac{C_i + C_e}{D} - g_*^{'}\psi_2 \equiv E_* (\psi_1).$$

- Equation for p_{i2}, p_{e2}

$$p_{j2} - p_{j1}{'}(\psi_1)\psi_2 = p_{j2*}(\psi_1) + \underline{\left(\frac{2x}{R_0}\right)\frac{C_j}{D}}, \quad (j = i, e),$$

- Ion stream function $\Psi \simeq \Psi_1 + \varepsilon\Psi_2, \quad n\mathbf{v} = \nabla\Psi \times \nabla\varphi + nRv_\varphi \nabla\varphi,$

$$\Psi_1^{'} = -\frac{n_0 R_0}{B_0}\left(\Phi_1^{'} + \frac{\lambda_H p_{i1}^{'}}{en_0}\right).$$

$$\Psi_2 = \Psi_{2*}(\psi_1) - \frac{n_0 R_0}{B_0}\Phi_{2*}(\psi_1) + \Psi_1^{'}\psi_2 - \underline{\frac{\lambda_H R_0}{eB_0 D}}\left[\underline{\frac{2x}{R_0}(C_i + C_e)} + P_{i2*} + P_{e2*}\right].$$

Ψ is not a function of magnetic flux in the presence of two-fluid effects.

- Grad-Shafranov (GS) equation for ψ_1

  - Identical to the static case

$$\Delta_2\psi_1 = -\mu_0 R_0^2\left\{\left(2x/R_0\right)\left[p_{i1}(\psi_1)+p_{e1}(\psi_1)\right]' + g_*'(\psi_1)\right\} - \left[I_1^2(\psi_1)/2\right]',$$

- GS equation for ψ_2

$$\Delta_2\psi_2 + \left\{\mu_0 R_0^2\left[\frac{2x}{R_0}p_1''(\psi_1)+g_*''(\psi_1)\right]+\left[\frac{I_1^2(\psi_1)}{2}\right]''\right\}\psi_2$$

$$= \frac{1}{R}\frac{\partial\psi_1}{\partial R} + \underline{F(\psi_1,\lambda_i,\lambda_H)}\Delta_2\psi_1 + \underline{F'(\psi_1,\lambda_i,\lambda_H)}\frac{\left|\nabla\psi_1\right|^2}{2} - \mu_0 R_0^2\left\{E_*'(\psi_1) - \frac{2x}{R_0}\left[\frac{P_{i2*}(\psi_1)+P_{e2*}(\psi_1)}{D}\right]'\right\}$$

$$-\mu_0 R_0^2\left(\frac{x}{R_0}\right)^2\left[p_{i1}(\psi_1)+p_{e1}(\psi_1)+2\frac{C_i(\psi_1,\lambda_i,\lambda_e,\lambda_H)+C_e(\psi_1,\lambda_i,\lambda_e,\lambda_H)}{D(\psi_1,\lambda_i,\lambda_e,\lambda_H)}\right]'$$

$$F(\psi_1) \equiv \left[V_E(\psi_1)-\lambda_H V_{di}(\psi_1)\right]\left[V_E(\psi_1)-\underbrace{(\lambda_H-\lambda_i)}_{\text{Gyroviscous cancellation}}V_{di}(\psi_1)\right], \quad \Delta_2 \equiv \left(\frac{\partial^2}{\partial R^2}+\frac{\partial^2}{\partial Z^2}\right)$$

$\boxed{\text{Single - fluid MHD : } (\lambda_i,\ \lambda_e,\ \lambda_H) = (0,0,0), \quad \text{Hall MHD : } (0,0,1), \quad \text{FLR two - fluid : } (1,1,1)}$

V_E, V_{di}, V_{de} : Poloidal Alfven Mach numbers of ExB, ion and electron diamagnetic drift velocity

Equilibrium with flow in reduced FLR two-fluid model - poloidal-sonic flow (2)

$$C_i = \beta_e \left[V_E + (\lambda_H - \lambda_i)V_{di}\right]\left\{(V_E + \lambda_H V_{di})\left[V_E - (\lambda_H - \lambda_e)V_{di}\right] - \lambda_e V_E V_{de}\left(1 - \frac{p_{e1}}{p_{e1}{}'}\frac{n_0{}'}{n_0}\right)\right\}\left\{\beta_i + \lambda_H V_{di}\left[V_E + (\lambda_H - \lambda_i)V_{di}\gamma\left(\frac{p_{i1}}{p_{i1}{}'}\frac{n_0{}'}{n_0}\right)\right]\right\}$$

$$-\beta_i \left\{(V_E + \lambda_H V_{di})\left[V_E + (\lambda_H - \lambda_i)V_{di}\right] + \lambda_i V_E V_{di}\left(1 - \frac{p_{i1}}{p_{i1}{}'}\frac{n_0{}'}{n_0}\right)\right\}$$

$$\times\left\{\beta_e \left[V_E - (\lambda_H - \lambda_e)V_{de}\right] - (V_E - \lambda_H V_{de})\left[V_E + (\lambda_H - \lambda_i)V_{di}\right]\left[V_E - (\lambda_H - \lambda_e)V_{de}\gamma\left(\frac{p_{e1}}{p_{e1}{}'}\frac{n_0{}'}{n_0}\right)\right]\right\}$$

$$C_e = \beta_e \left\{V_E\left[V_E + (\lambda_H - \lambda_i)V_{di}\gamma\left(\frac{p_{i1}}{p_{i1}{}'}\frac{n_0{}'}{n_0}\right)\right] - \beta_i\right\}\left[V_E + (\lambda_H - \lambda_i)V_{di}\right]\left\{(V_E + \lambda_H V_{di})\left[V_E - (\lambda_H - \lambda_e)V_{de}\right] - \lambda_e V_E V_{de}\left(1 - \frac{p_{e1}}{p_{e1}{}'}\frac{n_0{}'}{n_0}\right)\right\}$$

$$+\beta_i\beta_e \left[V_E - (\lambda_H - \lambda_e)V_{de}\right]\left\{(V_E + \lambda_H V_{di})\left[V_E + (\lambda_H - \lambda_i)V_{di}\right] + \lambda_i V_E V_{di}\left(1 - \frac{p_{i1}}{p_{i1}{}'}\frac{n_0{}'}{n_0}\right)\right\}$$

$$D = \left\{\beta_i - V_E\left[V_E + (\lambda_H - \lambda_i)V_{di}\gamma\left(\frac{p_{i1}}{p_{i1}{}'}\frac{n_0{}'}{n_0}\right)\right]\right\}(V_E - \lambda_H V_{de})\left[V_E + (\lambda_H - \lambda_i)V_{di}\right]\left[V_E - (\lambda_H - \lambda_e)V_{de}\gamma\left(\frac{p_{e1}}{p_{e1}{}'}\frac{n_0{}'}{n_0}\right)\right]$$

$$+\beta_e (V_E + \lambda_H V_{di})\left[V_E - (\lambda_H - \lambda_e)V_{de}\right]\left[V_E + (\lambda_H - \lambda_i)V_{di}\gamma\left(\frac{p_{i1}}{p_{i1}{}'}\frac{n_0{}'}{n_0}\right)\right]$$

$$\beta_i \equiv p_{i1}\Big/\left(B_0^2/\mu_0\right), \quad \beta_e \equiv p_{e1}\Big/\left(B_0^2/\mu_0\right)$$

- Single fluid $(\lambda_i, \lambda_e, \lambda_H) = (0,0,0)$

$$\frac{C_i + C_e}{D} = -\frac{V_E^2\left(\beta_i + \beta_e\right)}{V_E^2 - \left(\beta_i + \beta_e\right)}$$

Singular when poloidal flow velocity equals poloidal sound velocity.

- Hall MHD $(\lambda_i, \lambda_e, \lambda_H) = (0,0,1)$

$$\frac{C_i + C_e}{D} = -\frac{\left(V_E + V_{de}\right)^2\left(\beta_i + \beta_e\right)}{\left[V_E + V_{di}\gamma\left(\frac{p_{i1}}{p_{i1}'}\frac{n_0'}{n_0}\right)\right]\left[V_E - V_{de}\gamma\left(\frac{p_{e1}}{p_{e1}'}\frac{n_0'}{n_0}\right)\right] - \left(\beta_i + \beta_e\right)}$$

- FLR two-fluid $(\lambda_i, \lambda_e, \lambda_H) = (1,1,1)$

$$\frac{C_i + C_e}{D} = -\frac{V_E\left\{\beta_i\left(V_E - V_{de}\right)\left[V_E + V_{di} + V_{di}\left(1 - \frac{p_{i1}}{p_{i1}'}\frac{n_0'}{n_0}\right)\right] + \beta_e\left(V_E + V_{di}\right)\left[V_E + V_{di} - V_{de}\left(1 - \frac{p_{e1}}{p_{e1}'}\frac{n_0'}{n_0}\right)\right]\right\}}{\left[V_E^2 - \left(\beta_i + \beta_e\right)\right]\left(V_E - V_{de}\right) - \beta_e\left(V_{di} + V_{de}\right)}$$

Singularity is shifted by the Hall and FLR effects

$$\beta_i \equiv p_{i1}\Big/\left(B_0^2/\mu_0\right), \quad \beta_e \equiv p_{e1}\Big/\left(B_0^2/\mu_0\right)$$

Numerical analysis for two-fluid FLR equilibria with poloidal-sonic flow

- Linear profiles as in the single fluid case
- Density profile

 - Single-fluid MHD: appears only in $V_E = -\sqrt{\mu_0 m_i n_0}\left(R_0 \Phi_1{}' \big/ B_0\right)$

 - Hall MHD, FLR two-fluid: must be determined separately

 - Linear density profile $n_0 \equiv n_c \overline{\psi}_1$

$$V_E = V_{Ec} \overline{\psi}_1^{1/2}, V_{di} = V_{dc} p_{i1c} \overline{\psi}_1^{-1/2}, V_{di} = -V_{dc} p_{e1c} \overline{\psi}_1^{-1/2}$$

  - Hall MHD: convective term is singular at $\psi_1 = 0$
  - FLR two-fluid: singularity at $\psi_1 = 0$ is cancelled by the gyroviscosity

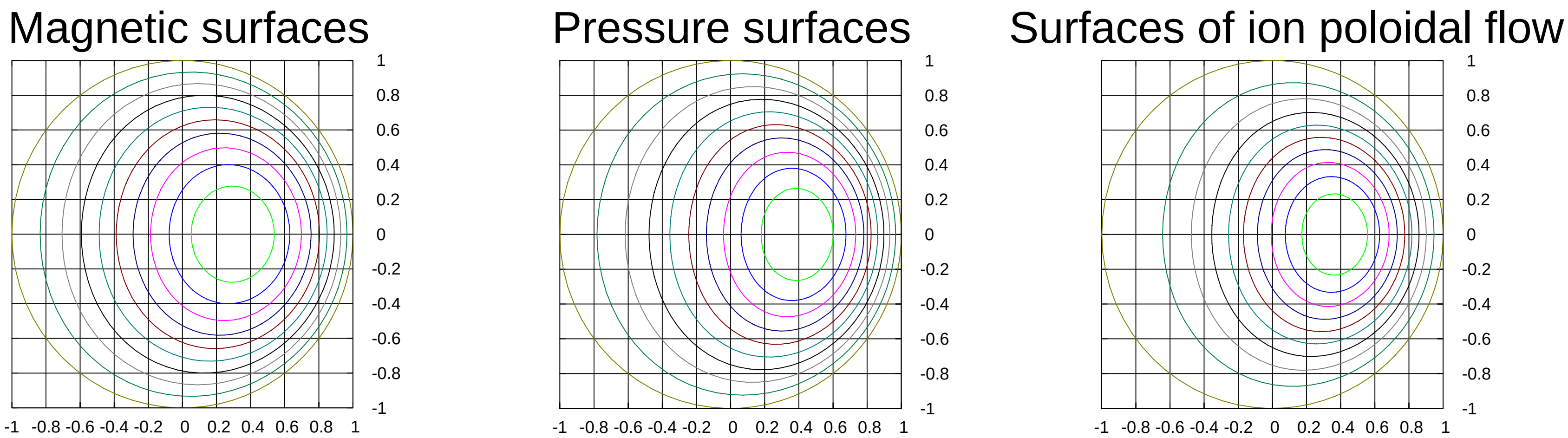
- Boundary conditions

$$\psi_1(1,\theta) = 0,\ \psi_2(1,\theta) = 0,\ p_2(1,\theta) = 0$$

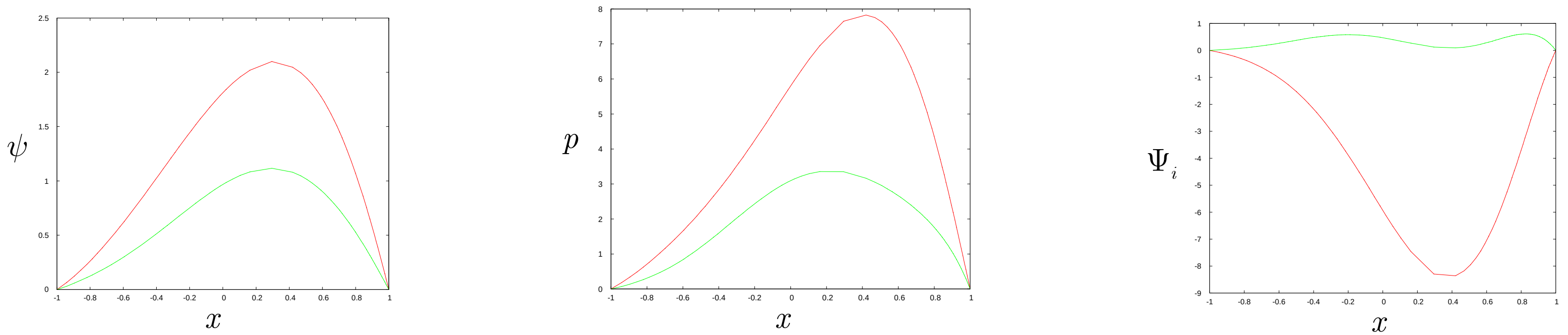
- Finite element method with 40×40 grids

Numerical solution for poloidal-sonic $E \times B$ flow

$$\left(V_{Ec}/\gamma\, p_{1c} = -1, V_{dc} = -1\right)$$



- Regular solution exists because the poloidal-sonic singularity is shifted by two-fluid and FLR effects.
- Pressure and ion poloidal-flow surfaces are not functions of magnetic flux.



Red: $V_{Ec}/\gamma\, p_{1c} = -1$ Green: $V_{Ec}/\gamma\, p_{1c} = 1$

- Solutions depends on the sign of ExB flow

- Shift of the maximum of Ion stream function $\overline{\Delta}_{\Psi} \simeq \overline{\Delta}_s + \varepsilon\overline{\Delta}_{\Psi 2}$

$$\overline{\Delta}_{\Psi 2} = -2\lambda_H V_{dc}\left[\frac{\overline{C}_i + \overline{C}_e}{\overline{D}}\left(\Psi_1' \frac{\partial^2 \overline{\psi}_1}{\partial^2 \bar{r}}\right)^{-1}\right]_{\bar{r}=\overline{\Delta}_{s1},\theta=0}$$

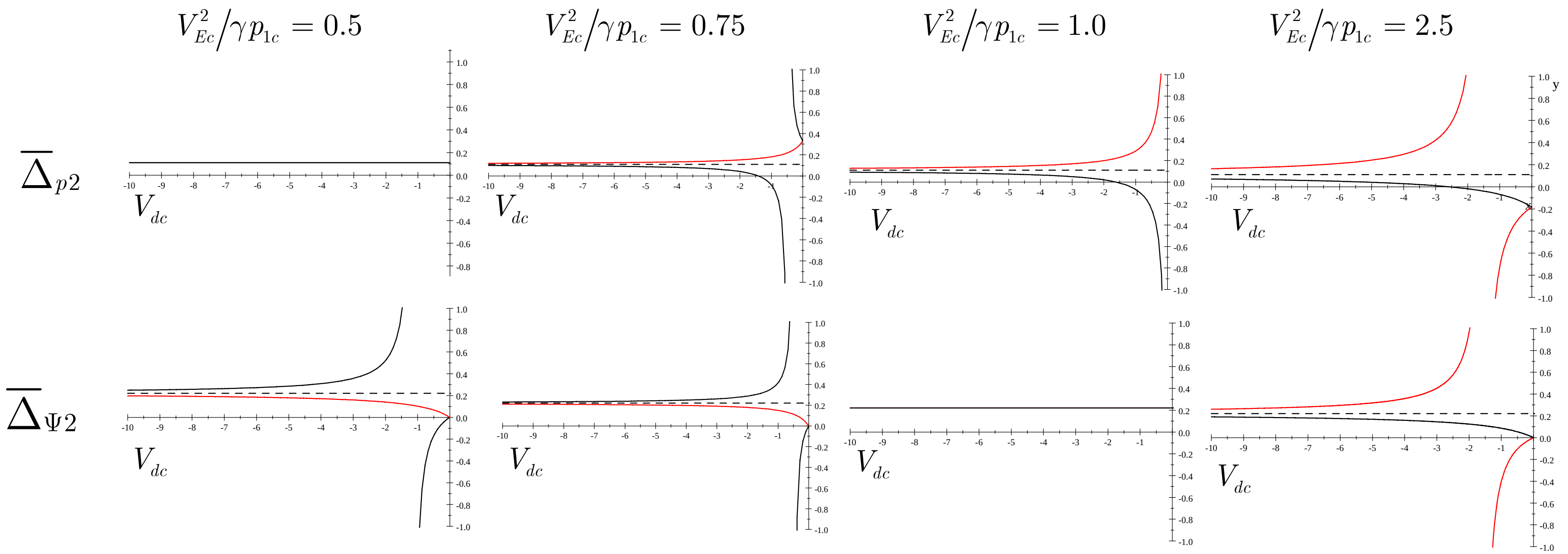
- Shift of the pressure maximum $\overline{\Delta}_p \simeq \overline{\Delta}_s + \varepsilon\overline{\Delta}_{p2}$

$$\overline{\Delta}_{p2} = -2\left[\frac{\overline{C}_i + \overline{C}_e}{\overline{D}}\left(\bar{p}_1' \frac{\partial^2 \overline{\psi}_1}{\partial^2 \bar{r}}\right)^{-1}\right]_{\bar{r}=\overline{\Delta}_{s1},\theta=0}$$

$$\sqrt{\varepsilon}V_{dc} \equiv -R_0\sqrt{\mu_0 m_i n_{0c}}\left(\varepsilon B_0^2 / \mu_0\right)\Big/ e n_{0c} B_0 \psi_c$$

$\overline{\Delta}_{\Psi 2}$ and $\overline{\Delta}_{p2}$ are functions of $\overline{\psi}_1$ and independent of $\overline{\psi}_2$

- Dependence of the shift of the maxima of the pressure and ion stream function on the diamagnetic drift for different values of ExB drift



Black: signs of ExB drift and ion diamagnetic drift are the same
Red: signs of ExB drift and ion diamagnetic drift are opposite

- $\overline{\Delta}_{p2}$ and $\overline{\Delta}_{\Psi 2}$ depends on the sign of ExB drift and the singularity shifted from the poloidal-sound velocity appears
- For $V_{Ec}^2/\gamma p_{1c} = 1.0$, $\overline{\Delta}_{\Psi 2}$ is independent of the diamagnetic drift
- For $V_{Ec}^2/\gamma p_{1c} = 0.5$, i.e. $V_{Ec}^2 = \gamma p_{i1c}$, $\overline{\Delta}_{p2}$ and $p_{i2} + p_{e2}$ are independent of the diamagnetic drift and analytic solution can be derived.

- Analytic solution can be found for $V_{Ec}^2 = \gamma p_{i1c}$

$$\psi_2 = P(r) + Q(r)\cos\theta + R(r)\cos 2\theta,$$

$$P(r) = \frac{1}{16}(r - 1^2)\left\{ -\left(\frac{p_{1c}}{B_p^2}\right)\left[r^2 + \frac{\gamma M_{Apc}^2}{\gamma p_{1c} - M_{Apc}^2}(r^2+1) \right] \right.$$

$$\left. + \frac{M_{Apc}^2}{8}\left[\frac{1}{9}\left(\frac{p_{1c}}{B_p^2}\right)^2 \left(13r^4 - 14r^2 - 5\right) + \left(\frac{g_c}{B_p^2}\right)^2 (3r^2 - 5) \right] \underline{-4V_{dc}\frac{g_c}{B_p^2}p_{1c}\sqrt{\gamma p_{1c}}} \right\}$$

$$Q(r) = -\frac{1}{16}r(r^2 - 1)\left\{ \frac{g_c}{B_p^2}\left[1 - \frac{M_{Apc}^2}{4}\frac{p_{1c}}{B_p^2}(3r^2 - 4) \right] \underline{-\frac{V_{dc}}{4}\frac{p_{1c}}{B_p^2}p_{1c}\sqrt{\gamma p_{1c}}} \right\}$$

$$R(r) = -\frac{1}{16}\frac{p_{1c}}{B_p^2}r^2(r^2 - 1)\left[1 + \frac{4}{3}\frac{\gamma M_{Apc}^2}{\gamma p_{1c} - M_{Apc}^2} - \frac{M_{Apc}^2}{48}\frac{p_{1c}}{B_p^2}(9r^2 - 11) \right]$$

- The Shafranov shift and the equilibrium beta-limit for this solution are independent of the diamagnetic drift

Summary (1)

We have obtained reduced equation for FLR two-fluid equilibria with flow in the order of poloidal Alfven and poloidal sound velocity.

- Poloidal-Alfvenic flow $v^2 \sim \varepsilon v_{th}^2 \sim v_{Ap}^2 \;\; \Rightarrow \delta^2 \sim \varepsilon$
  - Singularity appears in the lowest order equations for the magnetic flux.
  - Shift of Alfven singular point due to FLR effect has been found.

- Poloidal-sonic flow $v^2 \sim \left(B_p / B_0\right)^2 \gamma p / \rho \sim \varepsilon v_{Ap}^2 \;\; \Rightarrow \delta \sim \varepsilon$
  - Effects of compressibility, two-fluid and FLR are included in higher-order eqations.

Summary (2)

We have solved analytically for MHD and numerically for two-fluid FLR equilibria.

- Analytic MHD equilibria
  - The solution represents the modification of the magnetic flux and the departure of the pressure surfaces from the magnetic surfaces by the poloidal flow.
  - The pressure maximum shifts outward for a sub-poloidal-sonic flow and inward for a super-poloidal-sonic flow from the magnetic axis.

- Numerical analysis for two-fluid FLR equilibria
  - Regularized solution due to the two-fluid and FLR effects are found.
  - Pressure and ion poloidal-flow surfaces are not functions of magnetic flux.
  - Equilibrium depends on the sign of ExB drift