Theory and Application of Magnetic Self-Organization

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Toroidal Alternate for Fusion Component Testing

- Fusion Component Test Facility (CTF) complements ITER burning plasma experiment toward DEMO fusion power plant around 2050
- CTF Mission needs:
  - Volumetric neutron source which produces ~100 dpa (high fluence and duty factor)
  - Minimizes tritium consumption
- Toroidal alternates are attractive choices for CTF
  - Low aspect ratio minimizes surface to volume ratio (hence maximizes dpa/tritium consumption).
  - High beta leads to improved fluence/cost ratio.
- Spherical tokamak-based CTF is the front runner; Spheromak is also a compelling possibility.
- Absence of a central solenoid prevents inductive startup: magnetic self-organization can be used to produce the plasma-confining magnetic field.
Laboratory examples

- SSPX (LLNL)
- NSTX (PPPL, CHI)
- MST (Wisconsin)

- Electro-static drive: power supply → magnetic energy.
- Helicity injection allows flux conversion: toroidal ↔ poloidal.
Space and Astrophysical

Jet/Lobe

gravitation → orbital → magnetic → synchrotron radiation.

- Solar flare, extra-galactic jet and radio lobe (magnetic bubble).
- Footpoint motion drags field line around: kinetic (flow) energy into magnetic energy.
- Helicity injection allows flux conversion, a grand scale "dynamo" machine (most efficient?)
Outline

- **Background**
  - Taylor’s idea on helicity conserving magnetic relaxation.

- **How does magnetic relaxation work (dynamics of relaxation)**
  - Onset of primary instability: open field line kinks.
  - Triggering of secondary instability and helical instability cascade in space.
  - How far off is Taylor’s prediction.

- **How does magnetic self-organization work**
  - Self-organization: input energy goes into large scale structures
  - Magnetic self-organization is enabled by resonant coupling between energy drive and the driven-relaxed plasma.
    - The theory of driven oscillators

- **Resonant coupling in magnetic self-organization**
  - Linear resonant coupling:
    - unconstrained (Jensen-Chu, 1984) and constrained (Tang-Boozer, 2005)
  - Nonlinear resonant coupling: unconstrained and constrained (Tang, 2007)

- **Flux amplification and current multiplication**
  - Contrast between spheromak (unconstrained resonances) and spherical tokamak (constrained resonances)
  - Reactor implications on spheromak and spherical tokamak startup.

- **A new paradigm for astrophysical radio lobe magnetic fields**
  - Self-organization via driven-relaxation overcomes severe difficulties in MHD dynamo.
Taylor’s relaxation theory

What’s special about turbulencet dissipation?
- Magnetic energy dissipates fast $\propto \eta j^2 \sim l^{-2}$
- Magnetic helicity $\int A \cdot B d^3x$ dissipates slow $\propto \eta_j \cdot B \sim l^{-1}$
  - Faraday’s law $\partial B/\partial t = -\nabla \times \vec{E}$
  - Ohm’s law $\vec{E} = -\vec{v} \times \vec{B} + \eta j$

Taylor’s hypothesis (1974)
- Fast magnetic energy dissipation versus slow magnetic helicity dissipation leads to minimum magnetic energy state while conserving the magnetic helicity. (variational principle)

Taylor state
- Constrained minimization leads to $\vec{j} = \nabla \times \vec{B} = k\vec{B}$
- “k” is the Lagrangian multiplier.
Dynamical perspective on Taylor state

- Equation of motion for the plasma
  \[ \frac{\rho d\vec{v}}{dt} = -\nabla p + j \times B \]

- In long mean free path plasmas,
  \[ \vec{B} \cdot \nabla p = 0 \]

- Magnetic field line \( d\vec{x} / d\tau = \vec{B} \) is a 1 ½ degrees of freedom Hamiltonian problem, generally non-integrable. Stochastic field lines (ergodic) implies force-free,
  \[ j = \nabla \times B = kB \]

- \( \vec{B} \cdot \nabla k = 0 \) + magnetic field line ergodicity implies constant \( k \) globally, i.e. Taylor state.

- Force-free or nearly so is a far more robust concept than Taylor state.

How does relaxation work?

- Onset of primary instability: open field line kinks
- Triggering of secondary closed flux instabilities
- Helical instability cascade in space
Experiment
Fast camera view

Simulation (CHIP code: 512 CPU run)
mean poloidal flux; n=1 mode

NSTX CHI Experiment
Raman, et al
Onset of current-driven open field line kink

\[ \hat{V} \equiv \mu_0 V / \eta B_0 q_0 \]

Normalized Voltage:

\[ \hat{V} = 0.32 \quad \hat{V} = 0.64 \quad \hat{V} = 0.85 \]

Relaxation by helical instability cascade

\[ \frac{\partial \chi_0(\psi, t)}{\partial t} = V_D(\psi, t) + V_\eta(\psi, t) + V_\Phi(\psi, t) \]

(Tang and Boozer, Phys. Plasmas, 2004b)
Primary (open flux) and secondary (closed flux) modes

\begin{align*}
n=0 & \quad n=1 & \quad n=2 \\
\end{align*}
Time evolution of q profile during relaxation

(Tang and Boozer, Phys. Plasmas, 2004b)
Why magnetic self-organization

- Self-organization: energy goes into large scale structures
  - Wildly popular concept in science.
  - Laboratory plasmas: high field ST, spheromak, and RFP (reactor potential).
  - Astrophysical plasmas: origin of large scale magnetic fields (the flux problem).

- Resonant coupling is the physical mechanism that enables magnetic self-organization
Taylor state versus harmonic oscillators

- Taylor state: linear PDE
  \[ \nabla \times \vec{B} = k\vec{B} \]

- Homogeneous bdy condition
  - Eigenvalue problem.
    - No externally imposed vacuum magnetic field.

- Inhomogeneous bdy condition
  - Can be transformed into inhomogeneous linear PDE.
  - Driven problem.
    - With externally imposed vacuum magnetic field.

- Harmonic oscillator: linear ODE
  \[ \frac{d^2u}{dt^2} + \omega_0^2 u = 0 \]

- Driven harmonic oscillator:
  \[ \frac{d^2u}{dt^2} + \omega_0^2 u = f \sin \omega t \]

- Linear resonance
  \[ u = \frac{f}{\omega_0^2 - \omega^2} \sin \omega t \]

Wavelength \(\Leftrightarrow\) Frequency, but what about \(f\)?
Resonances in driven-relaxation

- Linear resonances in Taylor state
  - Jensen-Chu unconstrained resonance.
  - Simply connected: Spheromak and ST-PCC.
  - Tang-Boozer constrained resonance.
  - Doubly connected: RFP and ST-CHI.

- Nonlinear resonances in partially relaxed plasmas
  - Both constrained and unconstrained (Tang).
Unconstrained linear resonance

- Axisymmetry: \[ \vec{B} = G(\chi)\nabla \phi + \nabla \phi \times \nabla \chi \]
- G-S equation: \[ \Delta^* \chi + G \frac{dG}{d\chi} = \Delta^* \chi + k^2 \chi = 0 \]
- Solution: \[ \chi = \chi_v + \sum \alpha_i \chi_i \]
- Vacuum field: \[ \Delta^* \chi_v = 0, \chi_v |_{\partial\Omega} = \chi |_{\partial\Omega} \]
- CK modes: \[ \Delta^* \chi_i + k_i^2 \chi_i = 0, \chi_i |_{\partial\Omega} = 0. \]
- Jensen-Chu resonances (1984):

\[ \alpha_i = \frac{k^2}{k_i^2 - k^2} \langle \chi_v \chi_i \rangle \]

(Tang-Boozer, PRL, 2005a)

\[ \langle \chi_v \chi_i \rangle = 0 \rightarrow \text{Taylor's "mixed" state} \]
Constrained linear resonance

- Finite net toroidal flux $\psi_t$ constraint.

- In a torus:
  \[ G(\chi) = G_0 - k\chi \]
  \[ \Delta^* \chi - k(G_0 - k\chi) = 0 \]
  \[ G_0 = \frac{\psi_t + \iint \frac{k\chi}{R} \, dS}{\iint \frac{1}{R} \, dS} \]
  \[ (k^2 - k_j^2)\alpha_j - \sum_i k^2 \chi_i \langle \chi_j \rangle \alpha_i = k\psi_t \langle \chi_j \rangle + k^2 \chi_v \langle \chi_j \rangle - k^2 \langle \chi_v \chi_j \rangle \]

Complication arises because:

\[ \langle \chi_j \rangle \neq 0 \quad \chi_j \neq 0 \]

(Tang-Boozer, PRL, 2005b)
Constrained Resonance

\[(k^2 - k_j^2) \alpha_j - \sum_i k^2 \chi_i \langle \chi_i \rangle \alpha_i = k \psi_t \langle \chi_j \rangle + k^2 \chi_j \langle \chi_j \rangle - k^2 \langle \chi_i \chi_i \rangle\]

\(\langle B^2 \rangle\)

- **Black**: \(G_0 = 0\)
- **Red (ST)**: \(\psi_t = \text{const.}\)
- **Green (RFP)**: \(\psi_t = \text{const.}; \vec{B} \cdot \vec{n} = 0\)

(Tang-Boozer, PRL, 2005b)
Nonlinear regularization

\[ k(\chi) = k_c (1 - \varepsilon \frac{\chi}{\chi_{\text{inj}}}) \]

- around \( k_1 \)

\[ \chi = \chi_v + \alpha_1 \chi_1; \quad \alpha_1 = \sqrt{\frac{2}{3} \frac{\langle \chi_v \chi_1 \rangle \chi_{\text{inj}}}{\langle \chi_1^3 \rangle \epsilon}} \]

- away from \( k_1 \)

\[ \chi = \chi_v \mid \alpha_1 \chi_1; \quad \alpha_1 = \frac{2}{3} \frac{k_c^2 - k_1^2}{k_c^2} \frac{\langle \chi_1^2 \rangle \chi_{\text{inj}}}{\langle \chi_1^3 \rangle \epsilon}. \]

- Corresponding current profile is extremely hollow

\[ k'(\chi) = k_c \left( 1 - \frac{k_c^2 - k_1^2}{k_c^2} \frac{\langle \chi_1^2 \rangle}{\langle \chi_1^3 \rangle} \chi_1 \right). \]

- Further relaxation suggests

\[ k'(\chi) = k_c \left( 1 - \varepsilon \frac{\chi}{\chi_o} \right). \]

*(Tang-Boozer, PRL, 2005a)*
Nonlinear resonance

\[ k(\chi) = k_c(1 - \epsilon \chi / \chi_a). \]

(Tang, PRL, 2007)
Properties of nonlinear resonance

\[ k_c^r = k_1 \left| c' k_1 \frac{\langle \chi_1^3 \rangle}{2 \langle \chi_1^2 \rangle \langle \chi_1 \rangle} \right. \]

\[ \alpha_1 = -\frac{b}{a} = \frac{k_c^2}{\frac{k_c^2}{k_c^2 - (k_c^r)^2}} \frac{\langle \chi v \chi_1 \rangle}{\langle \chi_1^2 \rangle} \]
Flux amplification and current multiplication

- Reactor requirements
- Spherical tokamak reactor solenoid-free startup
- Spheromak reactor scenario
Reactor requirements

- Current multiplication factor (CM)
  - Ratio of toroidal plasma current vs. injector current
    \[ \mathcal{M}_C \equiv \frac{(I_{p}^{int} + I_{p}^{inj})}{I_{inj}}. \]

- Flux amplification factor (FA)
  - Ratio of interior poloidal flux vs. injector flux
    \[ \mathcal{A}_F \equiv \frac{\chi_{int}}{\chi_{inj}}. \]

- Reactor requirements:
  - High CM good for electrode power handling and reduces impurity production.
  - High FA implies high field, hence improved engineering efficiency.
  - For spheromak, CM & FA are linearly correlated.
  - For ST, CM & FA are independent or anti-correlated.
ST and spheromak have opposite FA-CM relations

- Consider the two-scale model: 
  - Sustained CHI discharge has 
    \[ k_{\text{inj}} < k_{\text{int}} \]
- For Spheromak, CM is linearly proportional to FA
  \[ \mathcal{M}_C = \tilde{q}_{\text{sp}} \left( 1 + \frac{k_{\text{clo}}}{k_{\text{inj}} A_F} \right) \]
  - Example: FA=6, average q=0.5, injector k twice the spheromak k, leads to CM=2, i.e. 10 MA spheromak has 5 MA going to the electrodes: unacceptable
- For spherical tokamak, CM is linearly, anti-correlated with FA
  \[ \mathcal{M}^{\text{ST-CHI}}_C = M - \left( 1 - \frac{k_{\text{int}}}{k_{\text{inj}}} \right) \tilde{q}_{\text{st}} A_F \]
  - M is the vacuum toroidal/injector poloidal flux ratio.
  - \( \tilde{q}_{\text{st}} = \frac{\psi_{\text{int}}}{\chi_{\text{int}}} \) is the flux ratio of the ST.
  - Luckily, FA is bounded from above, comparable with M.
  - Eventual CM is a substantial fraction of M.
ST solenoid-free startup by CHI

- Characteristic parameter:
  \[ M \equiv \frac{\psi_t}{\chi_v^0} \]

- Edge field reversal:
  \[ k_r = k_1 - \delta; \delta \approx k_1 / M \]
  - RFP is a limiting case

- Upper bound on FA:
  \[ A_{F}^{\text{st}, I_e} < A_{F}^{\text{upper}} \approx M / \bar{q}_1 \]

- Elongation is a key parameter
  - 2-3 is the reactor operational regime.

(Tang and Boozer, Phys. Plasmas, 2005b)
Partial relaxation has small impact!

ST-CHI operates far below the actual constrained resonance

for a partially relaxed plasma with $k(\chi) = k_e(1 - \epsilon \chi / \chi_a)$

(Tang & Boozer, Phys. Plasmas, 2007)
Spheromak FA must operate near resonances

- Linear resonant FA requires closeness to Taylor state.
  - Difficult to access in laboratory plasmas.

- Nonlinear resonant FA is the preferred route to reactor
  - High FA potential in plasma far away from Taylor state.
  - Consistent with finite pressure spheromak.
    - Resonance even with finite pressure.
  - Average q for nonlinear resonant mode is higher in identical chamber, so CM is larger for the same FA.
  - Can be facilitated via auxiliary heating and current drive.

(Tang & Boozer, Phys. Plasmas, 2008)
Astrophysical Magnetic Fields

- Planetary, stellar, and disk magnetisms are believed to be the result of MHD dynamo
  - Flow energy -> magnetic energy

- Radio lobe magnetic fields are likely the result of self-organization by driven-relaxation
  - Gravitational energy -> accretion disk flow energy -> helicity injection -> self-organized radio lobe magnetic field
Observational motivations

> 5% of available energy in lobe-scale magnetic fields (Kronberg, et al)

Severe energy constraint:
gravitation $\rightarrow$ orbital $\rightarrow$ magnetic $\rightarrow$ synchrotron radiation.

Intra-Cluster Medium embedded
MHD dynamo exceeds the energy budget!
(gravitational energy $\rightarrow$ flow energy $\rightarrow$ magnetic field)

Most optimistic dynamo:
(1) Preferentially grows small scale B initially;
(2) Inverse cascading to large scale B;
(3) Accumulated dissipation $\gg B^2$

Physics challenge:
How to get the desired B spectrum without paying a ransom in power consumption?

Hint: accretion disk–radio lobe system similar to laboratory spheromak experiments

(brandenburg, 2001)
Magnetic self-organization can be the answer! (external magnetic energy + helicity $\rightarrow$ large scale B)

- Driven-relaxation:
  $$\vec{j} = \nabla \times \vec{B} = \lambda \vec{B}$$

- Linear and nonlinear resonances in driven-relaxation constrain the input energy into system scale B fields.
  - Extreme form of self-organization.

- Under-driven plasmas self-organize into system-scale magnetic fields as in Taylor relaxation.

- Over-driven plasma is the frontier of research.
  - Astrophysical radio lobe appears to be over-driven.
  - Does it self-organize? To what degree? Under what condition?
  - LANL studies include theory, simulation, astrophysical data analysis, and comparison with experiments.

Summary

- How does magnetic relaxation work:
  - helical instability cascade from primary open flux kink to closed flux modes.

- Why magnetic self-organization in driven-relaxation:
  - The result of resonant coupling between energy and helicity drive and the driven-relaxed plasma.
    - Similar to driven oscillators
  - Linear resonances: unconstrained (spheromak and radio lobe) and constrained (spherocal tokamak and RFP).
  - Nonlinear resonances in partially relaxed plasmas.

- Theory clarifies fundamental physics constraints on the applications of helicity injection in ST and spheromak
  - Compute the flux amplification and current multiplication.
  - Give the physics basis for ST reactor solenoid-free startup.
  - Give the physics basis for a spheromak reactor.

- Magnetic self-organization via driven-relaxation is a new paradigm for large scale astrophysical magnetic fields.
  - Radio lobes are driven oscillators in disguise.
  - Magnetic self-organization in over-driven systems is a frontier for future research.
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