Magnetic Coupling of the Toroidal Plasma with External Asymmetries

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Coupling:
Magnetic interaction of the plasma with the wall (vacuum vessel), correction coils, error field

What will be the plasma reaction to the applied magnetic perturbation?
The Problem

Toroidal Geometry:
Plasma – vacuum gap – wall – vacuum – …

Approximation:
sometimes – thin-wall

The goal:
Coupling of the inner and outer solutions for the magnetic perturbation
Existing approaches

Different areas (plasma, wall, vacuum) – Different solvers

Examples: MARS (MHD) + CARIDDI (wall currents)

DCON (MHD) + VALEN (wall currents)

The coupling/decoupling is also needed in IPEC (the Ideal Perturbed Equilibrium Code)

DCON (MHD) + coupling of $\xi$ to external field (vacuum)
We derive the equation covering the “coupling strategy” described in


This also solves a similar coupling problem from

the primary interest is in the response of the plasma to very small perturbations, i.e., \( b/B \approx 10^{-2} \) to \( 10^{-4} \), which can be calculated using the theory of perturbed equilibria.

The ideal plasma response to external magnetic perturbation can be **computed** with high accuracy by the **code** that constructs a relevant interface between the actual field and the external field on the control surface.
Solution scheme: Actual field $\Rightarrow$ external surface current $\Rightarrow$ required external field

Linear operators:

$\vec{b} \cdot \hat{n} = \hat{\Lambda}[\vec{K}^x]$  
$\vec{b}^x \cdot \hat{n} = \hat{L}[\vec{K}^x]$

finally

$\vec{b} \cdot \hat{n} = \hat{P}[\vec{b}^x \cdot \hat{n}]$  
$\hat{P} = \hat{\Lambda}\hat{L}^{-1}$. 
The CarMa code, a self-consistent coupling between the MHD code MARS-F and the 3D eddy currents code CARIDDI, is applied to ITER geometry for the evaluation of the effects of 3D conducting structures on Resistive Wall Modes (RWM) control.

**Coupling strategy: the CarMa code**

Main assumption: **plasma mass is neglected**
- good approximation if the time scale is much longer than Alfvén time (related to plasma mass)
- plasma response to given input is instantaneous

A (de-)**coupling surface** $S$ is chosen
- any surface in between plasma and conducting structures
- plasma-wall interaction is decoupled via $S$

The plasma (instantaneous) response to a given magnetic flux density perturbation on $S$ is computed as a plasma response matrix.
Coupling Between a 3-D Integral Eddy Current Formulation and a Linearized MHD Model for the Analysis of Resistive Wall Modes

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**Coupling strategy**

A surface is chosen, in between the plasma and the conducting structures.

The plasma response to a given magnetic perturbation is computed as a plasma response matrix.

The effect of 3D structures on plasma is evaluated by computing the magnetic flux density on S due to 3D currents.

The currents induced in the 3D structures by plasma are computed via an *equivalent surface current* distribution on S.
In the both cases the needed relations are:

\[ \hat{b} \cdot \hat{n} = \hat{P}[\hat{b}^x \cdot \hat{n}] \]

In both cases not only \( \mathbf{b} \cdot \mathbf{n} \), but \( \mathbf{b} \times \mathbf{n} \) on \( S \) is also known

“IPEC uses the displacement of the plasma boundary to determine a part of the perturbed magnetic field that is normal to the unperturbed plasma boundary and a part that is tangential to the plasma boundary”

“In particular, each time we find the tangential component of \( b \) on \( S_e \)” (A. Portone, et al., 2008)

In the both cases to solve the problem an additional (“external, equivalent, superficial”) surface current is introduced and calculated

This step is actually NOT necessary
### Starting equations

#### Plasma:
\[
\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \mathbf{j} \times \mathbf{B} + \mathbf{f}
\]

#### Wall:
\[
\mathbf{j} = \sigma \mathbf{E}
\]

#### All space:
- \( \text{rot} \, \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \)
- \( \text{div} \, \mathbf{B} = 0 \)
- \( \text{rot} \, \mathbf{B} = \mu \mathbf{j} \)  
  (\(\mu\) : permeability)

#### Outside the plasma:
- vacuum gap,
- wall (\(\mu = \text{const}\), \(\sigma = \text{const}\))
- correction coils, error field

#### Boundary conditions:
\[
\langle n \cdot \mathbf{B} \rangle = 0 \quad \langle n \times \mathbf{B} / \mu \rangle = 0
\]
**Definitions**

\[ B = B_0 + b(r, \zeta, z) \]

**Magnetic field: Unperturbed + perturbation**

\[ b = b_{\text{plasma}} + b_{\text{wall}} + b_{\text{external}} \]

**In vacuum** 

\[ b = \nabla \varphi \quad \text{with} \quad \nabla^2 \varphi = 0 \]

\[ \varphi = \varphi_{\text{plasma}} + \varphi_{\text{wall}} + \varphi_{\text{external}} \]

**with** \( \nabla^2 \varphi^i = 0 \) \text{ in different areas for different } \varphi^i
Formulation of the problem

Assume that equation

\[
\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \mathbf{j} \times \mathbf{B} + \mathbf{f}
\]

is solved with given disturbed boundary (inside a torus \( S \))

A relation between the total field and the external field is needed on the ‘control surface’ \( S \)

\[
\mathbf{B}^{out}(\mathbf{B}) = ?
\]

Here

\[
\mathbf{B} = \mathbf{B}^{out} + \mathbf{B}^{pl}
\]
General solution

Use Biot–Savart law

\[
\mathbf{B}^{pl} = \frac{\mu_0}{4\pi} \nabla \times \int_{\text{plasma}} \frac{\mathbf{j}(\mathbf{r}_{pl})}{|\mathbf{r} - \mathbf{r}_{pl}|} \, dV_{pl} = \frac{\mu_0}{4\pi} \int_{\text{plasma}} \mathbf{j}(\mathbf{r}_{pl}) \times \frac{\mathbf{r} - \mathbf{r}_{pl}}{|\mathbf{r} - \mathbf{r}_{pl}|^3} \, dV_{pl},
\]

integration over the plasma volume.

Here \( \mathbf{j} = \nabla \times \mathbf{B} / \mu_0 \) must be known from the solution of the perturbed equilibrium problem.

Can it be transformed into the surface integral over the control surface (in general, not the plasma surface)?
Volume integral $\rightarrow$ control surface

1. Use the identity

$$\mu_0 \int_V (j \times \nabla f) dV = \int_S \left\{ \left( \mathbf{n} \times \mathbf{B} \right) \times \nabla f + \nabla f (\mathbf{n} \cdot \mathbf{B}) \right\} dS - \int_V \mathbf{B} \nabla^2 f dV$$

where $\mathbf{n}$ is the unit outward normal to $S$.

2. With $f = (4\pi |\mathbf{r} - \mathbf{r}'|)^{-1}$, which satisfies

$$\nabla^2 f(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}')$$

$$\mathbf{B}^{pl}(\mathbf{r}) = \frac{1}{4\pi} \int_S \left\{ \left( \mathbf{n} \times \mathbf{B} \right) \times \frac{\mathbf{r} - \mathbf{r}_s}{|\mathbf{r} - \mathbf{r}_s|^3} + (\mathbf{n} \cdot \mathbf{B}) \frac{\mathbf{r} - \mathbf{r}_s}{|\mathbf{r} - \mathbf{r}_s|^3} \right\} dS + \nu \mathbf{B}(\mathbf{r})$$

where

$$\nu = \int_V \delta(\mathbf{r} - \mathbf{r}') dV' = \begin{cases} 1 & \text{if } \mathbf{r} \in V \\ 0.5 & \text{if } \mathbf{r} \in S \\ 0 & \text{if } \mathbf{r} \notin V \cup S \end{cases}$$
Maxwell equations are linear, \( B = B_0 + b \), then

\[
b_{\text{out}} = b - b^{pl} = b(1 - \nu) - \frac{1}{4\pi} \int_S \left\{ (n \times b) \times \frac{r - r_s}{|r - r_s|^3} + (n \cdot b) \frac{r - r_s}{|r - r_s|^3} \right\} dS
\]

Or, outside the control surface \( S \):

\[
b^{pl}(r) = \frac{1}{4\pi} \int_S \left\{ (n \times b) \times \frac{r - r_s}{|r - r_s|^3} + (n \cdot b) \frac{r - r_s}{|r - r_s|^3} \right\} dS
\]

Can be used to find \( b^{pl} \) at the wall position
Compare to the CarMa coupling scheme

1. For some vacuum magnetic field \( b_v \) with \( n \cdot b_v = n \cdot b \) at \( S \):

\[
0 = \frac{1}{4\pi} \int_S \left\{ (n \times b_v) \times \frac{r - r_s}{|r - r_s|^3} + (n \cdot b_v) \frac{r - r_s}{|r - r_s|^3} \right\} dS + \nu b_v.
\]

2. Subtract from the equation for \( b^{pl} \), get the CarMa result:

\[
b^{pl}(r) = \frac{\mu_0}{4\pi} \int_S i \times \frac{r - r_s}{|r - r_s|^3} dS + \nu(b - b_v),
\]

where

\[
\mu_0 i = n \times (b - b_v)
\]
An alternative approach (1)

1. Start from

$$\nabla^2 \varphi^i = 0,$$

where $\varphi^i$ describe the contributions to $\varphi$.

$$\varphi = \varphi^{\text{plasma}} + \varphi^{\text{wall}} + \varphi^{\text{external}}$$

2. Use the Green’s second identity with $g = (4\pi|\mathbf{r} - \mathbf{r}'|)^{-1}$, which satisfies $\nabla^2 g(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}')$, to get

$$- \varphi^i(\mathbf{r}) \int \delta(\mathbf{r} - \mathbf{r}') dV' = \int (\varphi^i \nabla g - g \nabla \varphi^i) \cdot dS$$

3. Apply this for each $\varphi^i$ with different regions depending on $\varphi^i$.

An alternative approach (2)


\[
\varphi(\mathbf{r}) = \int (\varphi \nabla g - g \nabla \varphi) \cdot d\mathbf{S}_c - \int (\varphi \nabla g - g \nabla \varphi) \cdot d\mathbf{S}_w
\]

2. Our first separate equation:

\[
\varphi^{\text{plasma}}(\mathbf{r}) = \int (\varphi \nabla g - g \nabla \varphi) \cdot d\mathbf{S}_c
\]

3. Our second separate equation:

\[
\varphi^w(\mathbf{r}) + \varphi^{\text{ext}}(\mathbf{r}) = -\int (\varphi \nabla g - g \nabla \varphi) \cdot d\mathbf{S}_w
\]
It can be shown that

\[
\nabla \phi^{\text{plasma}} = \int_{\text{control}} \{ (n_c \times \nabla \phi) \times \nabla g + (n_c \cdot \nabla \phi) \nabla g \} dS_c
\]
Thin-wall equations (1)

\[ j = \sigma E, \quad \nabla \times E = -\frac{\partial B}{\partial t} \quad \Rightarrow \quad |\nabla a| \frac{\partial}{\partial t} b \cdot n_w = \nabla \cdot (\nabla \times j / \sigma) \]

where \( a(r) = \text{const} \) describes the toroidal surfaces:

- \( a = a_w \): the inner side of the wall (\( S_w \))
- \( a = a_w + da \): the inner outer side of the wall (\( S_{w+} \))

Assume that the wall is also thin \textit{magnetically}:

\[ n_w \cdot \nabla \varphi_w^- = n_w \cdot \nabla \varphi_w^+ \quad \text{Then} \quad j = \nabla \kappa \times \nabla H \]

where \( H(a - a_w) = H_w \) is the Heaviside step function
Thin-wall equations (2)

\[
\left| \nabla a \right| \frac{\partial}{\partial t} \mathbf{b} \cdot \mathbf{n}_w = \nabla \cdot (\nabla a \times \mathbf{j} / \sigma)
\]

\[\Rightarrow \quad \tau_w \frac{\partial}{\partial t} \mathbf{b} \cdot \mathbf{n}_w = \hat{l} \mu_0 \kappa
\]

where \( \tau_w = \mu_0 \sigma_0 r_0 d_0 \) is the ‘wall time’ with \( \sigma_0, r_0 \) and \( d_0 \) the constants representing ‘in average’ the conductivity, minor radius and thickness of the wall,

and \( \hat{l} \) is the operator defined on \( S_w \) by

\[
\hat{l}X \equiv r_0 \frac{d_w}{d_0} \nabla \cdot \frac{\sigma_0 d_0^2}{\sigma d_w^2} \left[ \mathbf{n}_w \times (\nabla X \times \mathbf{n}_w) \right]
\]
Thin-wall equations (3)

\[ j = \nabla \kappa \times \nabla H \quad \Rightarrow \quad \mathbf{b}^w = \frac{\mu_0}{4\pi} \int_{\text{wall}} (\nabla \kappa \times \mathbf{n}_w) \times \frac{\mathbf{r} - \mathbf{r}_w}{|\mathbf{r} - \mathbf{r}_w|^3} \, dS_w \]

Also, from \( j = \nabla \times \left( \frac{\mathbf{B}}{\mu} \right) = \nabla \times (\kappa \nabla H) \) we obtain

\[ \varphi^w_+ - \varphi^w_- = \mu_0 \kappa, \]  
and, after simple transformations,

\[ \varphi^w = -\mu_0 \int_{\text{wall}} \kappa \nabla g \cdot dS_w = -\tau_w \frac{\partial}{\partial t} \int_{\text{wall}} \hat{l}^{-1} (\mathbf{b} \cdot \mathbf{n}_w) \nabla g \cdot dS_w \]

Both representations are equivalent: \( \nabla \varphi^w = \mathbf{b}^w \)
In the cylindrical coordinates \((R, \zeta, z)\) with toroidal angle \(\zeta\):

\[
g \equiv \frac{1}{4\pi |r - r'|} = \frac{1}{4\pi^2 \sqrt{RR'}} \sum_{n=-\infty}^{\infty} e^{in(\zeta - \zeta')} Q_{n-1/2}(\chi),
\]

\[
Q_{n-1/2}(\chi) \equiv \frac{1}{2\sqrt{2}} \int_{0}^{2\pi} \frac{\cos nu}{\sqrt{\chi - \cos u}} du
\]

is the half-integer degree Legendre function of the second kind, \(\chi \equiv \frac{2}{k^2} - 1\) with

\[
k^2 \equiv \frac{4RR'}{(R + R')^2 + (z - z')^2}
\]

\[
Q_{n-1/2}(\chi) = i_n(k) + \ln \frac{k}{\sqrt{1 - k^2}}
\]

where \(i_n(k)\) is the function finite at \(k^2 = 1\).
**Approximations and applications (2)**

Axial symmetry of the integration surfaces,  
large-aspect ratio,  
circular plasma and wall  

\[
\tau_w \frac{\partial B_m}{\partial t} = \Gamma_m B_m - \Gamma^0_m B^{\text{ext}}_m
\]

which is the result of ‘cylindrical’ theory.

**Cylindrical approximation:**  
\[
b_r = \sum b_m(r,t) \exp(i m \theta - n \zeta)
\]

\[
b_m(r_w) = B_m = B^{\text{pl}}_m + B^{\text{wall}}_m + B^{\text{ext}}_m
\]

\[
\Gamma_m = -2m \frac{B^{\text{wall}}_m + B^{\text{ext}}_m}{B_m}
\]

also found from  
\[
\frac{rb'_m}{b_m} = -(\mu + 1) - \frac{2\mu \Gamma_m x^{2\mu}}{2\mu + \Gamma_m (1 - x^{2\mu})}
\]
Main equation

Equation for the mode amplitude at the wall:

\[ \tau_w \frac{\partial B_m^m}{\partial t} = \Gamma_m B_m + 2mB_{m}^{ext} \]

When \( B_m^{ext} = 0 \)

\[ \Gamma_m = \tau_w (\gamma_0 + in\Omega_0) \]

\( \gamma_0 \) is the natural growth/decay rate,

\( \Omega_0 \) is the natural toroidal rotation frequency of the mode.
Resonant Field Amplification

\[ \tau_w \frac{\partial B_m}{\partial t} = \Gamma_m B_m + 2mB_{m^\text{ext}} \]

At \( \Gamma_m = \text{const} \), \( \text{Re} \Gamma_m < 0 \), \( B_m = \text{const} \) (static perturbation)

\[ B_{m^\text{st}} = A B_{m^\text{ext}} \quad \text{with} \quad A = -\frac{2m}{\Gamma_m} \]

General geometry: \( \varphi = \varphi^{\text{out}} + \varphi^{\text{pl}} \),

Given \( \varphi^{\text{out}} \), find \( \varphi^{\text{pl}} \), calculate the ‘amplification’ ratio

\[ \varphi / \varphi^{\text{out}} = 1 + \varphi^{\text{pl}} / \varphi^{\text{out}} \]
Summary

- The mentioned problems are resolved analytically by an explicit expression. The general solution gives the necessary output at the same input as in CarMa.

- This shows, in particular, that a part of the CarMa numerical procedure is redundant and can be replaced by a precise analytical solution.

- This solves the Coupling problem as needed in the IPEC formulation (Park, et al., Phys. Plasmas 14, 052110 (2007)).

- The solution is ready for use, will be useful for any RWM code and in the “Perturbed equilibrium” problems.
For more details see

V.D. Pustovitov, **General formulation of the resistive wall mode coupling equations**, Phys. Plasmas 47, 072501 (2008);