Calculation of Stochasticity from Topological Noise in the DIII-D Shot 115467 3000 ms

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Abstract. An area-preserving map in magnetic coordinates is derived from Hamiltonian equations of motion for magnetic field lines using an infinitesimal canonical transformation of second type [1]. The map generating function for the field lines in the DIII-D is calculated from the experimental data for the shot 115467 at 3000 ms. The poloidal magnetic flux, χ, is the Hamiltonian for field lines. The equilibrium Hamiltonian function for the DIII-D, χ₀, is calculated from the shot data as a piece-wise defined function of toroidal flux, ψ. For 0≤ψ≤ψ₁, safety factor q increases monotonically to the value 5. For ψ₁≤ψ≤ψₕ, the safety factor increases logarithmically without limit. ψₕ is the toroidal flux inside separatrix in the DIII-D. The logarithmic singularity is symmetric about the separatrix. The singular region contains 5% of toroidal flux, and 0.87% of poloidal flux inside the separatrix in the DIII-D shot. In the open field line region outside the separatrix, q is defined by the distance a field line requires to go from its first to its second close approach to the X-point. In this region, the safety factor first decreases to the value 3.8, and then increases. Stochasticity caused by topological noise [2] in the DIII-D shot is calculated using this map. Topological noise consists of modes (m,n) = {(3,1), (4,1), (6,2), (7,2), (8,2), (9,3), (10,3), (11,3), (12,3)} with each amplitude equals to 0.8×10⁻⁵. Topological noise creates two very narrow layers of stochasticity. One is inside the separatrix and another is outside the separatrix. From the equilibrium data, a transformation from magnetic coordinates to the DIII-D (R,Z,φ) coordinates is calculated. This transformation is used to calculate stochasticity in physical space. Preliminary results of this investigation are presented. This work is supported by DE-FG02-01ER54624 and DE-FG02-04ER54793.

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I. Introduction

In this paper, we have attempted to make accurate calculation of the magnetic footprint on the target plate of the DIII-D divertor. The method used combines: (1) an accurate calculation of the rotational transform in magnetic coordinates from experimental data, (2) an accurate calculation of magnetic field line trajectories using...
a symplectic map with generating function obtained from (1), (3) an accurate calculation of physical coordinates \((R,Z)\) using a transformation from magnetic coordinates to physical coordinates. The transformation is determined from experimental data. We apply this approach to the DIII-D divertor tokamak shot 115467 at 3000 ms, and calculate the magnetic footprint on the target plate and stochasticity arising from the intrinsic topological noise [2].

### II. Derivation of the ICT Map

The relation between Hamiltonian mechanics and magnetic fields arises through the canonical representation of an arbitrary divergence-free vector, \(B(x)\). This representation uses four potentials \(\psi, \theta, \phi, \chi\) to describe an arbitrary divergence-free field,

\[
B = \nabla\psi \times \nabla\theta + \nabla\phi \times \nabla\chi. \tag{1}
\]

It is best to choose \(\phi\) as a toroidal angle and \(\theta\) as a poloidal angle. The potential \(\psi\) is then toroidal flux enclosed by a constant-\(\psi\) surface, and \(-\chi\) is poloidal flux outside a constant-\(\chi\) surface. The canonical form for \(B\), coupled with the chain rule for differentiation, yields Hamilton’s equations with \(\psi\) the canonical momentum, \(\theta\) the canonical coordinate, \(\phi\) the canonical time, and \(\chi(\psi, \theta, \phi)\) the Hamiltonian. An infinitesimal canonical transformation (ICT) with generating function \(F_2(\theta, \psi, \phi) = \theta + \epsilon \chi(\theta, \psi, \phi)\) [3] gives the ICT map

\[
\psi_{v+1} = \psi_v - \epsilon \frac{\partial \chi(\theta_v, \psi_{v+1}, \phi_v)}{\partial \theta_v}, \quad \theta_{v+1} = \theta_v + \epsilon \frac{\partial \chi(\theta_v, \psi_{v+1}, \phi_v)}{\partial \psi_{v+1}}. \tag{2}
\]

\(\epsilon\) is a small parameter. The determinant of the Jacobian matrix for this map equals unity, \(\frac{\partial(\psi_{v+1}, \theta_{v+1})}{\partial(\psi_v, \theta_v)} = +1\), so this map preserves area and orientation. The continuous analog of the map is also area-preserving. \(\epsilon\) is the map parameter, \(\epsilon = 2\pi/N_p\), where \(N_p\) is the number of iterations of map that is equivalent to a single circuit of tokamak. When the Hamiltonian for the field line is separable, \(\chi(\theta_v, \psi_{v+1}, \phi_v) = \chi_A(\psi_{v+1}) + \chi_B(\theta_v, \phi_v)\), the map becomes an explicit map.

### III. Calculation of the Hamiltonian from Data for the DIII-D Shot 115467 3000 ms

The generating function for the map is the equilibrium poloidal flux \(\chi_0(\psi)\). The total Hamiltonian is \(\chi(\psi, \theta, \phi) = \chi_0(\psi) + \chi_1(\psi, \theta, \phi)\). \(\chi_1(\psi, \theta, \phi)\) arises from magnetic perturbations, and breaks the toroidal symmetry. For the DIII-D shot 105467 at 3000
ms, we have calculated the equilibrium Hamiltonian as a piecewise defined, continuous function with continuous first and second derivatives, except at \( \psi = \psi_{\text{sep}} \) on the separatrix where the second derivative \( (d^2 \chi/d\psi^2)(\psi_{\text{sep}}) = \tau'(\psi_{\text{sep}}) \) is discontinuous. We divide the \( \chi \)-space into 4 regions. The first region is \( 0 \leq \chi \leq \chi_{b12} \). The second region is \( \chi_{b12} \leq \chi \leq \chi_{\text{sep}} \). In the second region, the safety factor \( q \) increases logarithmically with \( \psi \), and becomes singular on the separatrix surface. \( \chi_{b12} \) denotes \( \chi \) at the boundary of regions 1 and 2. \( \chi_{\text{sep}} \) denotes \( \chi \) on the separatrix surface. The third region is \( \chi_{\text{sep}} \leq \chi \leq \chi_{b34} \). In the third region the safety factor \( q \) decreases logarithmically with \( \psi \). The safety factor in the inner and outer logarithmic regions (regions 2 and 3) is symmetric about \( \psi_{\text{sep}} \). The fourth region is \( \chi \geq \chi_{b34} \). \( \chi_{b34} \) denotes \( \chi \) at the boundary of the region 3 and 4. In Fig. 1, we show the safety factor \( q(\psi) \) along with the experimental data. The logarithmic regions, regions 2 and 3, together contain 5% of toroidal flux and 1.74% of poloidal flux, of that inside the separatrix.

![FIGURE 1. Safety factor as function of toroidal flux, \( q(\chi) \), and the data points.](image1)

![FIGURE 2. Equilibrium surfaces from the fit and the equilibrium data for the DIII-D](image2)

The rotational transform \( \tau \) for the DIII-D is given by
where \( c_1 = 1.08874415148679, c_2 = -0.988849077883512, c_3 = 0.456609879025090, \)
\( c_4 = -9.943201006566417 \times 10^{-2}, d_1 = 1271.81065334683, d_2 = -1937.64199416408, \)
\( d_3 = 983.556636713237, d_4 = -166.310131671237, a = -0.381211434506300, \)
\( b = 3.89004273448613, \chi_{\text{sep}} = 1, \psi_{\text{sep}} = 1.8805476887122, \psi_{b12} = 1.83353399664947, \)
\( \chi_{b12} = 0.991314617567339, q_{b12} = 5.05552670348049, \psi_{b34} = 1.9275613810926, \)
\( \chi_{b34} = 1.00868538243276, q_{b34} = 5.05552670348078, \) and \( q_{b34}' = -4.21847664553362. \)

IV. Transformation of Canonical Variables (\( \psi, \theta \)) to DIII-D Coordinates (R,Z)

In Hamiltonian mechanics generalized coordinate and generalized momentum have equal footing and are independent. We first calculate \( \psi_{\text{data}} \) corresponding to \( \chi_{\text{data}} \) using the expressions for Hamiltonian \( \chi_0(\psi) \) obtained in the previous section. From experiment, \( \chi_{\text{data}} = (0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.85, 0.9, 0.95, 1.0, 1.02, 1.04). \) We choose the (R,Z) data two open surfaces (\( \chi = 1.02, 1.04, \psi = 1.97958952878253, 2.04743470385438 \)), the separatrix surface (\( \chi = 1, \psi = 1.8805476887122 \)), and the two closed surfaces closest to the separatrix (\( \chi = 0.9, 0.95, \psi = 1.45350456237793, 1.62699413299561 \)). We express these data as \( \psi(R-R_0, Z-Z_0) \), where \( (R_0, Z_0) = (1.758 \text{ m}, -0.023 \text{ m}) \) are the coordinates of the O-point. We fit a bivariate polynomial in (R-R_0) and (Z-Z_0) to fit \( \psi(R-R_0, Z-Z_0) \) for these five surfaces We use nonlinear regression for the fit. The best fit is obtained for 5th degree in R-R_0, and 6th degree in Z-Z_0,

\[
\psi = \sum_{i=1}^{5} A_i (R-R_0)^i + \sum_{j=1}^{6} B_j (Z-Z_0)^j + \sum_{i=1}^{5} C_{\theta i} (R-R_0)^i (Z-Z_0)^j \quad (4)
\]

with coefficients \( B_1 = 0.104229321412567, B_2 = 0.972534071321006, \)
\( B_3 = -0.312649411322068, B_4 = 2.6119663889582, B_5 = 0.276172236640846, \)
\( B_6 = -1.03078160705312, A_1 = 0.262904659626213, A_1 = 0.262904659626213, \)
\( C_{11} = 3.40425256107379 \times 10^{-2}, C_{12} = -0.80718237990807, \)
\[ C_{13} = -0.71397557864181, \quad C_{14} = 9.44663868816226, \quad C_{15} = 0.621173895457709, \]
\[ C_{16} = -4.04635657620473, \quad A_2 = 0.383793592135729, \quad C_{21} = 1.49722241935013, \]
\[ C_{25} = -3.6135377122972 \times 10^{-2}, \quad C_{26} = -3.20224629138651, \]
\[ A_3 = -0.868341528638289, \quad C_{31} = -0.844529937572852, \quad C_{32} = 28.5759574231828, \]
\[ C_{33} = 2.39497089464742, \quad C_{34} = -20.013064168369, \quad C_{35} = -2.67680380115931, \]
\[ C_{36} = -3.93512536235526, \quad A_4 = -18.3024509134956, \quad C_{41} = -0.92896587079174, \]
\[ C_{42} = 11.2817774360547, \quad C_{43} = 20.9175857390027, \quad C_{44} = -20.4224589757406, \]
\[ C_{45} = 34.043943724682, \quad A_5 = -14.703125173247, \quad C_{51} = -29.8986911055841, \]
\[ C_{52} = 32.0055762841356, \quad C_{53} = 18.582944121337, \quad C_{54} = -75.0349863630728, \]
\[ C_{55} = -34.01349863630728, \quad C_{56} = 29.505048895717. \]

In Fig. 2, we show the flux surfaces from data and fit.

If magnetic coordinates \((\psi, \theta)\) are known, we can calculate the physical angle \(\zeta(\psi, \theta) = \tan^{-1}\left(\frac{Z-Z_0}{R-R_0}\right)\) using a set of rules. Equation (4) can be transformed into an 11th degree polynomial in \(r = \sqrt{(R - R_0)^2 + (Z - Z_0)^2}\) with coefficients that are known functions of \(\zeta\). Solution of this polynomial gives \((R, Z)\).

**V. Stochasticity due to statistical topological noise in the DIII-D**

The spectrum of statistical topological noise [2] in the DIII-D is given by the locked modes \((m, n) = \{(3,1),(4,1),(6,2),(7,2),(8,2),(9,3),(10,3),(11,3),(12,3)\} \}

First we calculate the last good surface for topological noise. We start 10000 field lines inside the separatrix and advance each line for 10000 toroidal circuits of tokamak using the map with topological noise. Field lines are spaced equally in \(\psi\)-space between \(\psi = 1.5\) and \(\psi = \psi_{sep}\) with \(\theta_0 = \theta_X\) and \(\phi_0 = 0\) where \(\theta_X\) is the angular location of the X-point. So each field line starts on the line joining the O-point to the X-point in the principal plane of the DIII-D. After each iteration of the map, the \((R, Z)\) coordinates of the line are calculated. We determine the line closest to the O-point that does not become an open line in 10000 toroidal circuits of the DIII-D. We call the surface determined by this line the last good surface, and denote it by \(\psi_{LGS}\). We call the distance from the X-point to the last good surface along the line joining the X-point to the O-point, the width \(w\) of the layer of open lines in the DIII-D due to topological noise. We find that \(\psi_{LGS} = 1.88039546979567, w = 5.407656910119585E-003\) m, and width in \(\psi\) space, \(\Delta \psi = (\psi_{sep} - \psi_{LGS}) = 1.522910755485209 \times 10^{-4}\).

We find that the topological noise creates stochasticity only in region 4. Fig. 3 shows the effect of topological noise. In this figure, we have assumed periodicity in \(\theta\). However, a more appropriate approach will be to calculate some another quantity (such as Liapunov exponents) to determine the character of open field lines, or a map that does not give rise to magnetic monopole. We are working on this. The magnetic
footprint due to stochastic noise on the left plate is shown in Fig. 4. For the footprint, we start 100,000 field lines between $\psi_{LGS}$ and $\psi=2.10661848096077$. The starting $(\psi, \theta, \phi)$ are chosen randomly. Each line is advanced at most for 1,000 toroidal circuits of tokamak.

![FIGURE 3. Stochasticity due to topological noise in the DIII-D](image1)

![FIGURE 4. Magnetic footprint on left plate due to topological noise.](image2)

## VI. Conclusions

We have developed an accurate method to calculate field line trajectories and magnetic footprints in (R,Z) and magnetic coordinates in divertor tokamaks. This method is applied to calculate width of open lines region inside separatrix, stochasticity and footprint due to topological noise in DIII-D. We are refining the method. Method has good promise to apply to other tokamaks, especially ITER.

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## References