Effects of a sheared ion velocity on the linear stability of ITG modes

M. Lontano\textsuperscript{1}, M.C. Varischetti\textsuperscript{1,2}, E. Lazzaro\textsuperscript{1}

\textsuperscript{1}Istituto di Fisica del Plasma, C.N.R., Euratom-ENEA-CNR Association, Milan, Italy
\textsuperscript{2}Dipartimento di Fisica, Universita’ degli Studi di Milano, Milan, Italy

Abstract. The linear dispersion of the ion temperature gradient (ITG) modes, in the presence of a non uniform background ion velocity $U_i = U_i(x)$, in the direction of the sheared equilibrium magnetic field $B_0 = B_0(x)$, has been studied in the frame of the two-fluid guiding center approximation, in slab geometry. Generally speaking, the presence of an ion flow destabilizes the oscillations. The role of the excited K-H instability is discussed.

Keywords: ion temperature gradient (ITG) instabilities, Kelvin-Helmoltz instability, drift modes, sheared flow, quasi-linear fluxes, diffusion coefficient.

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INTRODUCTION

The presence of a spontaneous [1] or driven [2-4] plasma rotation in tokamak discharges is well established, as well as its principal role in the determination of the stability properties of different macroscopic and microscopic modes. Indeed, in the past years, several theoretical investigations have been carried out on the linear and non-linear dynamics of the ITG modes, when the background plasma develops an inhomogeneous macroscopic velocity, being this quite a general issue concerning space plasma physics [5], as well. With reference to a tokamak plasma configuration, a sheared poloidal rotation has a non-linearly stabilizing effect against modes producing radially elongated vortices [6]. The role of the toroidal component of the plasma rotation is however more intriguing, as it is shown from the results of several theoretical studies where either slab [7-13] or toroidal [14-16] ITG stability issues have been investigated. In addition, the stability properties of the ITG modes with a toroidal sheared velocity are of particular interest as a first principle approach to the determination of the anomalous momentum transport observed in tokamak plasmas.

In this paper, we extend our previous studies [17] of the dispersive properties of the ITG modes in a uniform field $B_0 = B_0\hat{b}$, in the presence of a sheared ion velocity $U_i = U_i\hat{b}$, to a non-uniform magnetic field $B_0 = B_0(x)\hat{b}(x)$, with curvature and shear [18]. The effects of field non-uniformity and field line curvature are modeled by introducing a gravitation-like drift velocity $v_{LG} = -\frac{1}{\Omega_i}\hat{b} \times \mathbf{g}$, where $\Omega_i = qB_0/Mc$ and...
\[ g = \frac{w_{\perp} + 2w_{\parallel}}{MB_0} \nabla B_0. \] Here, \( q_i = Ze \) and \( M \) are the electric charge and the mass of the ion species, \( w_{\parallel(\perp)} = \frac{M}{2} v_{\parallel(\perp)}^2 = \frac{p_i}{n_i} \) is the parallel (perpendicular) microscopic kinetic energy of the ions. In the following two sections we shall describe (i) the details of the physical model to derive the 3rd degree linear dispersion relation, and (ii) the results of the analysis of the dispersion properties of the ITG modes in different plasma regimes.

**THE MODEL**

We consider a plasma slab in the \( x \)-direction, with complete uniformity in the \( y,z \) plane. At the equilibrium, the magnetic field \( \mathbf{B}_0 = \mathbf{B}_T + \mathbf{B}_p \) and plasma velocity \( \mathbf{U}_i = U_i \hat{\mathbf{b}} \) along \( \mathbf{B}_0 \) are allowed to vary along \( x \), together with all other plasma parameters, \( n_s, T_s, p_s \), with \( s = e,i \). For low frequency perturbations, with \( \omega \ll \Omega_i \), the perpendicular dynamics of the ions can be described by a fluid drift velocity

\[
v_{\perp} = \frac{c}{B} \mathbf{b} \times \nabla \phi + \frac{c}{q_i n_i B} \mathbf{b} \times \nabla p_i + \frac{c}{q_i n_i B} \hat{\mathbf{b}} \times \nabla \cdot \Pi + \frac{1}{\Omega_i} \hat{\mathbf{b}} \times \frac{\nabla v_{\perp}}{dt} + v_g; \]

here, the electric drift, the diamagnetic drift, the anisotropy pressure drift, the polarization drift and the magnetic drift terms, respectively, are recognizable in the r.h.s. Due to the long time-scales considered, electrons are able to achieve thermal equilibration along the magnetic field lines, so that the longitudinal distribution of the perturbed electron density can be assumed to satisfy the Boltzmann distribution \( \tilde{n}_e(\mathbf{r},t) = n_s(x) \exp(\mathbf{e} \phi/T_s) \). Quasi-neutrality \( \tilde{n}_e = \tilde{Z} \tilde{n}_i \) is assumed, as well. Ion density, ion parallel velocity and ion pressure are described by means of the continuity equation, the parallel momentum equation, and the adiabatic equation, respectively. Gyroviscosity is retained in the momentum equation, while both collisional and collisionless dissipation mechanisms are neglected. Kinetic effects can be safely neglected, as far as \( \omega/k_i v_i >> 1 \). Due to the magnetic field \( \mathbf{B}_0 \) non-uniformity, the unitary vector \( \hat{\mathbf{b}} \) changes direction moving along \( x \). Therefore, for the sake of simplicity, at any \( x \) a local Cartesian coordinate system is introduced with \( \hat{e}_z = \hat{\mathbf{b}} \).

**The dispersion relation.** Under the above positions, the fluid equations can be worked out and, after a Fourier analysis in \( y,z,t \), we are left with the “local” dispersion relation

\[
\tau^3 + a_2 \tau^2 + a_1 \tau + a_0 = 0
\]

where

\[
a_2 = -\left(1 - \frac{\Gamma \tau}{Z}\right) \omega_{sne} + \tau(2 + \Gamma) k_z c_s
\]

\[
a_1 = -Z k_z c_s^2 \left(1 + \frac{\tau}{Z} \frac{1}{k_z \Omega_i} \right) - \frac{\Gamma \tau}{Z} \omega_{sne}^2 +
\]

\[
+r c_s \left(2 + \Gamma \left(1 - \frac{\tau}{Z}\right) \omega_{sne} + Z \left(1 + \Gamma \frac{\tau}{Z}\right) \omega_{s}\right) + r \tau^2 (1 + 2\Gamma) k_z^2 c_s^2
\]

\[\tau \equiv k_z c_s / \omega_{sne} \]
\[ a_0 = k_i^2 c_i^2 \left[ Z \left( 1 + \frac{\Gamma \tau}{Z} \right) \omega_{s_i} + \Gamma \tau \frac{k_i^2 U_s^*}{k_i \Omega} \right] - r \tau k_i c_i^2 \left[ \frac{\Gamma \tau}{Z} \omega_{s\! ne}^2 + Z k_i^2 c_i^2 \left( 1 + \frac{\Gamma \tau}{Z} \frac{k_i^2 U_s^*}{k_i \Omega} \right) \right] - r^2 \tau k_i^2 c_i^2 \left[ \tau (1 + \Gamma) \omega_{s\! ne} + Z \left( 1 + \frac{\Gamma \tau}{Z} \omega_{s_i} \right) + r^3 \Gamma \tau^3 k_i^2 c_i^3 \right] \]

(4)

Here, \( \omega = \omega - k_i U_i(x) \) is the Doppler shifted frequency of the mode; \( k_i \parallel \) and \( k_i \perp \) are the parallel and perpendicular components of the wave-vector referred to the direction of \( B_0 \) at the position \( x \); \( \Gamma = 5/3 \) is the adiabatic index, \( \tau = T_e / T_i \), \( c_s = \sqrt{T_e / M} \) is the sound velocity, \( \rho_i = c_i / \Omega_i \) is the ion Larmor radius at the sound velocity; \( \omega_{s_i} = \frac{k_i c_i^2}{\Omega} \) and \( \omega_{s\! ne} = \frac{k_i c_i^2}{e B_0} (\ln n_i)' \) are the ion and electron diamagnetic frequencies, respectively, which are calculated with the unperturbed parameters; finally, \( r = 3 \rho_i / 2 L_B \) and \( L_B^{-1} = (\ln B_0)' \). The apex indicates the total differentiation with respect to \( x \).

**FIGURE 1.** The lines \( \gamma = 0 \) (marginal stability) in the plane \( [R/L_{ni}] \mid [R/L_{Ti}] \) (left plot, a), for different values of \( R/L_{ni} \), and in the plane \( [R/L_{ni}] \mid [R/L_{Ti}] \) (right plot, b), for different values of \( R/L_{ni} \).

Eq.(1) has been solved by assuming simple analytical expressions for the \( x \)-profiles of the various physical parameters: \( n_i(x) = n_i(1 - x^2/a^2)^\alpha \), \( T_i(x) = T_i(1 - x^2/a^2)^\beta \), \( U_s(x) = U_s(1 - x^2/a^2)^\delta \), with \( s = e, i, \) and \( a \) is the slab half thickness. Moreover, \( B_T(x) \) is taken \( \propto R^{-1} \) (with reference to a corresponding tokamak configuration) and \( B_p \) is calculated from a given \( q(x) \) profile. As it is seen from Eqs.(3,4), the presence of
\( U'_b \neq 0 \) introduces a dependence of the dispersion on the sign of \( k_\parallel \) [13,17]. Specifically, modes with \((k_\perp/k_b)U'_b(\parallel) > 0\) are (de-)stabilized. Therefore, directing our attention to \( k_\perp > 0 \), for \( U_b(x) \) profiles peaked on-axis, the most unstable modes have \( k_\parallel < 0 \). Unless explicitly mentioned, we shall consider these modes in the present analysis. According to the scaling of ITG modes, i.e. \( |k| \ll k_\parallel \), we take \( |k| \approx 10^{-3}k_\perp \).

The quasi-linear fluxes. On the basis of the linear equations that relates the perturbed velocities \( \tilde{v}_x, \tilde{v}_y, \tilde{v}_n \) to \( \tilde{\phi} \), we can calculate the quasi-linear (q.l.) fluxes along \( x \) of the parallel momentum \( \Gamma_{\tilde{v}_x} = M_n \langle \tilde{v}_{x,\tilde{v}_x} \rangle \) of and the ion pressure \( \Gamma_p = \langle \tilde{v}_{x,\tilde{v}_i} \rangle = n_i \langle \tilde{v}_{x,\tilde{T}_i} \rangle \). The spatial average (in \( y \) and \( z \)) of two generic perturbed scalar quantities \( \tilde{A}(\mathbf{r}, \mathbf{t}) \) and \( \tilde{B}(\mathbf{r}, \mathbf{t}) \) is defined as

\[
\langle \tilde{A} \tilde{B} \rangle = \lim_{L \to \infty} \frac{1}{L^2} \int_{-L/2}^{+L/2} \int_{-L/2}^{+L/2} d\mathbf{y} \int_{-\infty}^{\infty} dk_\perp \int_{-\infty}^{\infty} dk_\parallel \text{Re} \{ \tilde{A}_k(\mathbf{x}) \tilde{B}_k^*(\mathbf{x}) \} \exp(2\gamma_k t) .
\]

The spatial interval \( L \) over which the average is performed formally does not appear when the electrostatic energy density \( \langle \tilde{\phi}^2 \rangle = \lim_{L \to \infty} \int_{-L/2}^{+L/2} \int_{-L/2}^{+L/2} d\mathbf{y} \int_{-\infty}^{\infty} dk_\perp \int_{-\infty}^{\infty} dk_\parallel |\tilde{\phi}_k|^2 \exp(2\gamma_k t) \) is introduced, once the fluctuation spectrum \( |\tilde{\phi}_k|^2 \) is specified. For example, for a simple bi-chromatic spectrum made of two vectors with the same \( \perp \) component, \( k_\parallel \neq 0 \), and opposite \( \parallel \) components, \( \pm k_\parallel \), \( |\tilde{\phi}_k|^2 = \delta(k_\perp - k_0)[I_\perp \delta(k_\parallel - k_0) + I_\parallel \delta(k_\parallel + k_0)] \), we get \( \langle \tilde{\phi}^2 \rangle = \langle \tilde{\phi}_\perp^2 \rangle + \langle \tilde{\phi}_{\parallel}^2 \rangle \), where \( \langle \tilde{\phi}_\parallel^2 \rangle = (1/2\pi L^2) \exp(2\gamma_{k_\parallel} t) \), so that the e.s. energy density can be written in terms of the intensities \( I_{\parallel} \). Being a linear theory we can’t find a saturation value for the turbulence, so that the fluxes will be normalized to \( e^2 I_{\perp}/T_{e,\perp} \). Moreover, it is convenient to present the fluxes in units of their typical dimensional values: \( \Gamma_{\tilde{v}_x} = n_i M_0 c T_e / e B k_\perp c_s \), for the flux of parallel momentum, and \( \Gamma_p = (c T_e / e B) k_\perp n_i T_i \), for the flux of ion thermal energy.

THE RESULTS

The case of a uniform magnetic field. First we discuss the stability of ITG modes with \( B_0 = B_0 \hat{z} \), with \( B_0 = \text{const.} \), that is, described by Eqs.(1-4) where \( L_b \to \infty \), implying \( r = 0 \). In Fig.1a, the red (continuous) line refers to \( U'_b = 0 \), that is, to basic ITG modes. Indeed, for a fixed value of \( |R/L_{n_i}| \), they are destabilized with increasing \( |R/L_{T_i}| \). If a non-zero velocity shear is introduced the picture is strongly modified. An \( U_b(x) \) peaked on-axis (\( U'_b < 0 \)) introduces a region of instability to the right of the red line, which is localized at low \( |R/L_{T_i}| \), for small velocity shear, but extends to high values of \( |R/L_{T_i}| \) with increasing \( |R/L_{U_b}| \). Here \( R/L_{U_b} = U'_b / \Omega_i \). Besides the appearance of the new unstable region, the red line moves to the left indicating a slight stabilization of the original (for \( U'_b = 0 \)) modes. On the contrary, a reversed \( U_b(x) \) profile (\( R/L_{U_b} > 0 \)) has the only effect to destabilize the original modes.
FIGURE 2. The lines \( \gamma_k = 0 \) in the plane \[ R/L_{U_i} \] (left plot, a), for different values of \( \tau \) (at \( T_e = \text{const.} \)), and in the plane \[ R/L_{T_i} \] (at \( T_i = \text{const.} \)) (right plot, b), for different values of \( R/L_{n_i} \).

In Fig.2, the role of the ion-to-electron temperature ratio \( \tau \) is analyzed. In the left plot the lines correspond to different \( \tau \) values obtained taking \( T_e \) fixed and varying \( T_i \), the full (black) line being \( \tau = 1 \). The instability induced by \( R/L_{U_i} < 0 \) is stabilized by increasing \( \tau \). However at very low \( \tau \), a new unstable region appears, affecting mainly \( R/L_{U_i} > 0 \) and higher \( R/L_{T_i} \). In the right plot the lines correspond to different \( R/L_{n_i} \) values, \( R/L_{T_i} = -7.5 \), and \( \tau \) varies at constant \( T_i \). Instability takes place mostly for \( \tau \leq 1 \), while the increase of \( \tau \) reduces the unstable region.

FIGURE 3. The lines \( \gamma_k = 0 \) as in Fig.1a, at the high field side (left plot, a), with \( B_0 = 2.4T \) and \( R/L_B = -1.17 \), and at the low field side (right plot, b), with \( B_0 = 1.7T \) and \( R/L_B = -0.87 \).
Figs.1,2 suggest that the presence of $U'_\parallel < 0$ excites a Kelvin-Helmholtz (K-H) instability [7], which manifests itself in the new unstable region at relatively low values of $\left| R/L_{τ_1} \right|$. Outside this region, the velocity shear has a slight stabilizing effect. On the other side, $U'_\parallel > 0$ destabilizes the ITG modes for very flat density profiles and $τ < 1$. High $τ$ values are beneficial to both ITG and K-H instabilities.

The case of a non-uniform magnetic field. Let’s consider Eqs.(1-4) with $r ≠ 0$. Then the relevant dispersion refers to ITG modes in the presence of the non-uniform magnetic field intensity $B_0(x)$, and of the “toroidal” curvature of field lines, according to the $v_{\perp G}$ drift. Magnetic shear is not included, here. In Fig.3 it is shown how the parameter space $\left| R/L_n \right| \left| R/L_{τ_1} \right|$ is modified, with respect to Fig.1a, when the magnetic curvature and inhomogeneity are taken into account. In the high field side (at $x=-0.5a$, left plot) the whole unstable region is compressed towards lower $\left| R/L_{τ_1} \right|$ values, while in the low field side (at $x=+0.5a$, right plot) both ITG modes and K-H instabilities are strengthened, the higher $\left| R/L_{τ_1} \right|$ values being dominated by K-H for $U'_\parallel < 0$.

![Diagram](image)

**FIGURE 4.** The lines $γ_L=0$ as in Fig.3b, when magnetic shear is added. In the inserts, the corresponding $q(x)$ profiles and $x$-positions. A detailed explanation is reported in the text.

The effect of magnetic shear. The shearless slab model with the “toroidal” field only is enriched by adding a “poloidal” component of the magnetic field on the basis of a given $q(x)$ profile. Then, the typical shear-scale $L_{s}=q(dq/dx)^{-1}$ is introduced. The effect of the shear on the dispersion relation can be described according to two different schemes. First, let’s consider a given wave-vector $(k_\perp,k_\parallel)$ with $|k_\parallel|<<|k_\perp|$, such that ion Landau damping is negligible. A sheared magnetic field makes the components $k_\perp$ and $k_\parallel$ vary in such a way that the mode will be localized in a finite $x$-interval, being strongly damped outside it. Therefore, if we compare the marginal stability lines ($γ_k=0$) in the shearless case [$B_r(x)$ only] with those corresponding to the
case with sheared magnetic field \([\mathbf{B}_p(x)\neq0]\), in general the modes will be stabilized, just because of the rotation of \(\mathbf{B}_0(x)\) with respect to \(\mathbf{k}\), in the same \(x\)-position.

However, with the new orientation of \(\mathbf{B}_0\), new \(\mathbf{k}\)’s will be made unstable, satisfying the inequality \(|k|<|k_1|\). From this point of view, the stabilizing effect of the magnetic shear refers to a given \(\mathbf{k}\) vector. In addition to this effect, there is a second intrinsic modification of the dispersion arising when a \(\mathbf{B}_p(x)\) is added, that is new terms proportional to \(d\theta_s/dx\) appear in the dispersion, where \(\theta_s = \tan^{-1}(\mathbf{B}_p/\mathbf{B}_T)\) is the pitch angle of the magnetic field lines. In our model these terms turn out to be negligibly small, although stabilizing. In Fig.4 the lines \(\gamma_k=0\) are drawn in the same parameter space as in Fig.3b (low field side), for two \(q(x)\) profiles: a monotonously increasing one (left plot, \(a\)) and a hollow one (right plot, \(b\), for modes with \(k_0<0\). The curves labeled with \(R/L_{U_i} = 0.016\) (group I) and \(-0.016\) (group III) refer to the case of purely toroidal magnetic field. The unstable region for the given \((k_x,k_y)\) is identified. Then, the \(\mathbf{B}_p\) component corresponding to the given \(q\)-profile, at \(x=0.25m\) and \(x=0.15m\), respectively, is introduced and the stability of the same \((k_x,k_y)\) is analyzed. The group of curves II (“with \(\mathbf{B}_p\)”) shows a strong stabilizing effect at \(R/L_{U_i} \leq 5\). However, if only the \(\theta_1\) terms are retained (“with \(\theta_1\)”), then \(R/L_{U_i} < 0\) has a strong destabilizing effect (group III), while \(R/L_{U_i} > 0\) does not affect appreciably the stability (group I).

![Graph showing the q.l. momentum flux \(\Gamma_U\) and ion pressure flux \(\Gamma_p\) as functions of the slab coordinate \(x\), for different density, temperature and velocity profiles.](image)

**FIGURE 5.** The q.l. momentum flux \(\Gamma_U\) (left plot, \(a\)) and ion pressure flux \(\Gamma_p\) (right plot, \(b\)), across the magnetic field, as functions of the slab coordinate \(x\), for different density, temperature and velocity profiles. The “*” indicates that the momentum profile is reversed. Here, \(a=0.5m\) and \(R=1.65m\).

**The quasi-linear fluxes.** In Fig.5 the dimensionless q.l. ion momentum flux \(\Gamma_U\) and the ion pressure flux \(\Gamma_p\), normalized to \((e^2 I_x/T_e^2) \Gamma_{U,x}\) and to \((e^2 I_x/T_e^2) \Gamma_{P,x}\), respectively, are plotted as functions of \(x\), in the low field side portion of the slab. Several density, temperatures and velocity profiles have been considered, labeled by \(\alpha,\beta,\gamma\). Centrally peaked velocity distributions cause outward fluxes of momentum and pressure. If an inverted velocity profile (that is, hollow) is considered (curve *), the momentum flux reverses, while the pressure flux is almost unaffected.
SUMMARY

The analysis of the linear stability properties of the electrostatic modes, which are excited in a plasma slab in the presence of a sheared ion flow, parallel to the equilibrium magnetic field, show that: i) modes with \((k_{⊥}/k_{||})U'_{||} > 0\) are destabilized by the shear flow [13,17]; ii) this is due to a new instability appearing at relatively low values of \(R/L_T\), which can be interpreted as the K-H instability already predicted by Catto et al. [7]; iii) an increasing value of \(T_i/T_e > 1\) has a stabilizing effect; iv) if density profiles are very flat, then modes with \((k_{⊥}/k_{||})U'_{||} < 0\) are also destabilized; v) the magnetic shear is stabilizing for a given wave-vector, competing with the effect of \(U'_{||} ≠ 0\), however new modes can be locally excited; vi) q.l. momentum flux show a variety of spatial profiles depending on the equilibrium configuration.

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