Tokamak equilibria with toroidal current reversal: properties and computational issues

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Abstract. Several properties of axisymmetric plasma equilibria with toroidal-current reversal (TCR) are discussed using some unifying concepts from catastrophe theory. Namely, those of structural stability of functions near critical points, singularity unfolding by small perturbations, and model parameter-space division by bifurcation sets are found to be of particular usefulness. Magnetic configurations displaying, simultaneously, TCR and nested flux surfaces are thence shown to be necessarily degenerate and structurally unstable, meaning that they are easily transformed into non-nested ones by small perturbations in the model parameter set. This should lead to a new paradigm when discussing TCR equilibria, as most of present knowledge relies mainly on the properties of nested solutions, which is expected to favor the study of the broader class of non-nested configurations that recently attracted a considerable discussion in the fusion community. In addition, it is also shown how TCR imposes some constraints on plasma profiles, and how these may be dealt with computationally while keeping the ability to manipulate the shape of the inner island system.

INTRODUCTION

Reversed shear discharges, with hollow toroidal current-density profiles, are known to enhance energy confinement and reduce plasma transport by developing internal transport barriers (ITB) [1, 2], turning them into suitable candidates for advanced tokamak scenarios. Following attempts to achieve extreme reversed discharges, configurations with nearly zero current density throughout a significant region around the plasma core (the so-called current hole) where reported [3, 4]. Quite significantly, it was not possible to find experimental evidences of toroidal current reversal [5], where the current density would decrease down into negative values somewhere inside the current hole. Indeed, in recent experiments, current drive was forced inside the hole after it has been established and stabilized, and it was unsuccessfully both in co- and counter-direction [6]. All this collection of experimental evidences seems to point towards the fact that magnetic equilibria with toroidal current reversal (TCR) are not feasible, prompting a still ongoing discussion.

Before more complex questions are made about TCR configurations, like their susceptibility to magnetohydrodynamic (MHD) instabilities or their possible evolution into magnetic reconnection scenarios, the single issue of their existence as solutions of the Grad-Shafranov (GS) equation should be firmly established, together with techniques allowing the latter to be effectively solved. To this end, the current knowledge about TCR equilibria is briefly reviewed in the next section, exposing a fundamental division between nested and non-nested configurations. Next, some concepts taken from the field of catastrophe theory are used to understand the reported properties and, more important,
to rule out nested configurations as isolated degenerate cases, while non-nested ones are found to be the most general forms of TCR equilibria. Finally, a few examples will be given and dully solved.

**A BRIEF REVIEW**

One of the most striking features of TCR equilibria is the existence of a poloidal-field reversal (PFR) layer, a closed line in the poloidal cross section along which the tangential magnetic field must vanish and which encloses a zero toroidal current. Since field lines are bound to specific flux surfaces, only two arrangements are possible: either flux surfaces are all nested around the magnetic axis, with the PFR layer matching one of them (in which case the magnetic field has toroidal component only and $\nabla \psi = 0$), or it must be orthogonal to every flux surface it crosses, with an inner island system unfolding around it (Fig. 1).

At first glance, it seems impossible to keep a PFR layer inside a plasma equilibrium because a non-zero poloidal field is needed to balance the hoop force leading to the Shafranov shift [7]. Assuming nested flux surfaces, this shift $\Delta$ is given by

$$\frac{1}{\rho} \frac{d\Delta}{d\rho} = \frac{\rho - (\rho + B^\theta B_\theta)}{B^\theta B_\theta},$$  \hspace{1cm} (1)$$

where $\rho$ is a flux label and $\theta$ an angle measured over each flux surface, $\rho(\rho)$ is the plasma pressure, $B^\alpha$ and $B_\alpha$ are the contravariant and covariant components of the magnetic-field vector

$$\mathbf{B} = \nabla \phi \times \nabla \Psi + B_\phi \nabla \phi,$$  \hspace{1cm} (2)$$
and $\langle f(r) \rangle$ denote the usual averaging $\int_0^r uf(u)du$. Looking at Eq. (1), it appears that a vanishing $\nabla \psi$ leads to a singular $\Delta$, thus breaking the equilibrium condition.

However, the previous argument is easily circumvented by letting the numerator approach zero over the PFR layer $\rho_L$ at a faster or equal rate than $B^\phi B_\phi$ [8]. But if this is so, then

$$p(\rho_L) = \langle p(\rho_L) \rangle + B^\phi(\rho_L)B_\phi(\rho_L),$$

and the pressure there must be greater than its space average. Moreover, differentiating the previous equation with respect to $\rho$, the derivative $dp/d\rho$ is found to vanish at the PFR layer. Taken together, these two facts suggest that only some special hollow pressure profiles are allowed, as confirmed later using a version of the virial theorem [9]. That the pressure profile needs to be hollow is a mere consequence of $p(\psi)$ being assumed to be a smooth function of the non-monotonic poloidal flux, while the restriction imposed on $\dot{p}(\psi) = dp/d\psi$ over $\rho_L$ does not have any natural justification. This is worsened when ellipticity and triangularity are taken into account, as the equations for each of them display singular quotients similar to the one in Eq. (1), which can only be handled by assigning specific values to the boundary conditions. As any slight perturbation of such conditions would break the equilibria, these were termed isolated and not feasible experimentally.

Recent attempts to lift some of these constraints involved taking into account plasma rotation effects, with the pressure ceasing to be a flux function and therefore enabling monotonic profiles to be considered in nested TCR equilibria [10]. Nonetheless, the requirements on specific boundary conditions could not be relaxed and it was argued that small variations in these could be balanced by corresponding changes in other parameters of the equilibrium model, so the singularities could remain under control [10]. This idea points to the existence of a set of relations between model parameters, which should remain invariant as the equilibrium is changed. The deeper implications of this idea will become clear later on in this discussion.

On the other hand, significant developments were also taken using non-nested configurations, with GS solutions displaying TCR being successfully computed for a wide variety of input profiles. These included flat pressure and current density [11], linear GS source terms of the form $J^\phi(R,\psi) = R^2 a + b + a \psi$ (with $a$, $b$, and $\alpha$ constants) [12, 13, 14], and full nonlinear models $J^\phi(R,\psi)$ with only a few restrictions in its zeroth-order (with respect to an inverse aspect-ratio expansion) component [15]. Albeit hollow, as a consequence of following the non-monotonic shape of $\psi(R,z)$, pressure profiles were not subject to any other constrains, the same being true for boundary conditions also, and the resulting equilibria were not found to be isolated. The pragmatic justification for this is that Shafranov shift, ellipticity, triangularity, and other flux-surface shape parameters cannot be defined in non-nested configurations, avoiding in this way the need to handle the singularities displayed by their defining equations. However, this seems to draw a more fundamental distinction between the two kinds of configurations, whose statement and discussion provides the subject for the following sections.
FIGURE 2. Contour levels of the degenerate forms $-u^2 - v^3$, $-u^2 - v^4$, and $-u^3 - y^5$ (from left to right).

ELEMENTS OF CATASTROPHE THEORY

One of catastrophe theory goals is to classify the local shape (or topology) of smooth functions and the way these behave when the parameters on which they depend are varied [16, 17, 18]. To keep the discussion as simple as possible, let $\psi(x, y)$ be a smooth bivariate function and $(\bar{x}, \bar{y})$ a point of its domain. Then, after a suitable smooth and invertible change of coordinates with origin in $(\bar{x}, \bar{y})$ defined by $u(x, y)$ and $v(x, y)$, the function $\psi(x, y)$ is reduced either to the plane

$$\psi(u, v) = u$$

(4a)

if $\partial_l \psi(\bar{x}, \bar{y}) \neq 0$, to a nondegenerate quadratic form

$$\psi(u, v) = \lambda_u u^2 + \lambda_v v^2$$

(4b)

if $\partial_l \psi(\bar{x}, \bar{y}) = 0$ and $\lambda_u \lambda_v \neq 0$, where $\lambda_u$ and $\lambda_v$ are the eigenvalues of the Hessian matrix $\partial_{ij} \psi(\bar{x}, \bar{y})$, or to one of the degenerate forms

$$\psi(u, v) = \lambda_u u^2 + f(v), \quad \psi(u, v) = f(u) + \lambda_v v^2, \quad \text{or} \quad \psi(u, v) = f(u, v),$$

(4c)

the first if $\lambda_v = 0$ only, the second if $\lambda_u = 0$ only, and the last if both vanish, all of them with $\partial_l f(0) = 0$ and $\partial_{ij}^2 f(0) = 0$. For smooth functions $\psi(x, y)$, most of the points in their domain are regular (i.e., $\partial_l \psi \neq 0$) and locally around them the contour levels look like parallel lines. However, the overall disposition of the latter is utterly determined by the location and type of the critical ones, where the gradient of $\psi(x, y)$ is zero. If at these critical points the Hessian is not singular, they are termed Morse (or nondegenerate) critical points and give rise either to o-points ($\lambda_u \lambda_v > 0$) or x-points ($\lambda_u \lambda_v < 0$). In turn, degenerate critical points have an ambiguous role in contour-level organization, yielding o-points or cusp-like lines (Fig. 2).
Usually, the function $\psi(x,y;c)$ under study is a solution of a certain equation which may involve several parameters $c_1, \ldots, c_m$, as is indeed the present case, with the GS equation depending on input-profile model parameters and boundary conditions. So, it is quite natural to inquire how a small perturbation $\Delta c$ in the values of the parameters $c$ does change the topology of $\psi(x,y;c)$. To answer this question, one starts by noting that the new solution $\psi(x,y;c + \Delta c)$ differs from the original one by a small perturbation, that is,

$$\psi(x,y;c + \Delta c) = \psi(x,y;c) + \Delta c \cdot \nabla_c \psi(x,y;c) + \cdots,$$

if everything depends smoothly on the parameters $c$. Therefore, the question is now to find out how each one of the canonical forms (or shapes) listed above changes when a small arbitrary perturbation is added to it, an issue for which catastrophe theory provides precise tools.

The first part of the answer to the problem in question is given by the concept of structural stability of nondegenerate forms: adding a small perturbation to a function does not qualitatively change its topology in the vicinity of regular or Morse critical points. More rigorously, a new change of coordinates $u'(u,v)$ and $v'(u,v)$ may always be found, after adding the perturbation, which enables the previous canonical shape, i.e. $\psi(u',v',c + \Delta c) = u'$ or $\psi(u',v';c + \Delta) = \lambda_{uv} u'^2 + \lambda_{v'} v'^2$, to be recovered [17]. On the contrary, degenerate forms are structurally unstable, meaning that any slight perturbation added induce a qualitative change in the function topology near a degenerate critical point. As it also turns out, the result of every perturbation added to a degenerate form $f(u,v;c)$ (sometimes termed a catastrophe germ) may be reduced to one of several canonical forms (termed as catastrophe unfoldings)

$$f(u,v;c + \Delta c) = f(u,v;c) + \sum_{i=1}^{\mu-1} \alpha_i(\Delta c) \varphi_i(u,v)$$

depending only on the reduced set of $\mu - 1$ parameters $\alpha(\Delta c)$, with $\alpha(c) = 0$, where $\mu$ is the so-called Milnor index and the $\varphi_i(u,v)$ are special monomials of the degenerate variables only [18].

Although the study and systematic classification of unfoldings lies at the very core of catastrophe theory, it is their effect on the local topology that is relevant for the current discussion. And these effects, in general, reduce themselves to the simple annihilation of the degenerate critical point, which is turned into a regular one, or its transformation into one or more Morse critical points (the so-called morsification process). This is clearly illustrated in Fig. 3 for the unfolding

$$f(u;\alpha) = u^3 - \alpha u$$

of the most simple germ $u^3$ (the elementary fold catastrophe), whose degenerate critical point at the origin is turned into the Morse critical points located at $u = \pm \sqrt{\alpha}/3$, for $\alpha > 0$, or into a regular point for $\alpha < 0$.

When handling more complex germs, with Milnor index $\mu > 2$ and whose unfoldings depend thus on more than one perturbation parameter [Eq. (6)], then a perturbation outcome can be something different from a regularization or a morsification process,
with the critical point remaining degenerate, but only under certain conditions. As an example, consider the cusp unfolding

\[ f(u; \alpha) = u^4 + \alpha_2 u^2 + \alpha_1 u, \]

with its critical points located at the solutions of \( 4u^3 + 2\alpha_2 u + \alpha_1 = 0 \). Some of these are Morse critical points, but those solving also the equation \( 12u^2 + 2\alpha_2 = 0 \) must be degenerate. Requiring both conditions to hold yields the relation

\[ \Xi(\alpha_1, \alpha_2) = 27\alpha_1^2 + 8\alpha_2^3, \]

which defines a surface in parameter space called the bifurcation set [17, 18]. Only for parameters on this measure-zero set does a perturbed form remain degenerate, and crossing it leads to a sudden change in the topology, with Morse critical points being created or annihilated (Fig. 4).
FIGURE 4. Bifurcation set \( \Xi(\alpha_1, \alpha_2) \) of the cusp unfolding (a) and topological changes (b) for the three points in parameter space \( \alpha_1 = -\frac{1}{2} \) (dotted line), \( \alpha_1 = -1 \) (solid line), and \( \alpha_1 = -\frac{1}{2} \) (dashed line), all with \( \alpha_2 = -\frac{1}{2} \).

NESTED EQUILIBRIA AS A MEASURE-ZERO SET

With the concepts developed in the previous section, it is now possible to draw some conclusions about the nature of nested and non-nested configurations. Looking back at Fig. 1, one easily finds that every point of the nested PFR layer is a degenerate critical point. That these points must be critical, with the gradient \( \nabla \psi \) vanishing throughout the layer, was already stated before. On the other hand, since they are disposed along a continuous line and are therefore nonisolated, they must also be degenerate. And why must a nonisolated critical point be necessarily degenerate? Because, at a nondegenerate Morse critical point, the canonical form in Eq. (4b) do apply and equating its gradient to zero yields always an isolated solution at the origin.

With a whole line of degenerate critical points, nested configurations are highly susceptible to the effects of small perturbations in the values of model parameters. And now, one knows what to expect from such effects: annihilation of most of the degenerate critical points and a mosification process to a few (or, more rigorously, a discrete set) of them, leading to non-nested equilibria displaying an inner island system and to nested topology destruction. Of course, it is also possible to keep a nested equilibrium invariant under perturbations, but then the values for these must be taken along the bifurcation set only and must obey relations similar to the one in Eq. (9). This recovers the idea of conditioned parameter variation [10], mentioned in the previous brief review. However, because the bifurcation set is a surface (or a set of surfaces) embedded in the higher dimension parameter space, the absence of any physical principle restricting perturbations to lie on it means that, in general, arbitrary perturbations lead to non-nested configurations. Ultimately, and considering a smooth mapping from parameter space into the space of GS equation solutions, nested equilibria with TCR are parametrized by a measure-zero set (the bifurcation set) and make thus a measure-zero set also. So, non-nested equilibria are the more representative class of TCR solutions of the GS equation.
The GS equation
\[-R^2 \nabla \cdot (R^{-2} \nabla \psi) = -J_\phi(R, \psi) = R^2 \dot{p}(\psi) + \dot{Y}(\psi), \tag{10}\]
is herein solved using a perturbative approach (where \(J_\phi(R, \psi), p(\psi), \) and \(Y(\psi)\) are the normalized covariant toroidal current density, the plasma pressure, and the squared poloidal current) [19], which has been adapted to cope with the existence of an internal PFR layer [15]. In brief, the solution \(\psi(r, \theta)\) is sought in laboratory coordinates \(r\) and \(\theta\) (with \(r\) the radial distance to the axis normalized to the minor radius \(a\) and \(\theta\) a poloidal angle measured clockwise from the equatorial plane at the high-field side), avoiding the need to define a mapping to some set of flux coordinates and, consequently, the need to make any assumptions about flux-surface topology. It works by expanding every function in powers of the inverse aspect ratio \(\varepsilon = a/R_0\) (with \(R_0\) the major radius),
\[R(r; \theta; \varepsilon) = 1 - \varepsilon r \cos \theta \tag{11a}\]
\[\psi(r; \theta; \varepsilon) = \psi_0(r) + \sum_{n=1}^{+\infty} \sum_{k=0}^{n} \varepsilon^n \psi_{nk}(r) \cos k \theta, \tag{11b}\]
\[p[\psi(r, \theta; \varepsilon)] = p_0[\psi_0(r)] + \varepsilon p_0[\psi_0(r)](\psi_{10} + \psi_{11} \cos \theta) + \cdots, \tag{11c}\]
\[Y[\psi(r, \theta; \varepsilon)] = Y_0[\psi_0(r)] + \varepsilon Y_0[\psi_0(r)](\psi_{10} + \psi_{11} \cos \theta) + \cdots, \tag{11d}\]
insert them into Eq. (10), and collect afterwards for the same powers of \(\varepsilon\). This procedure is known to yield a single nonlinear equation for the zeroth-order \(\psi_0(r)\),
\[r^2 \psi''_0(r) + r \psi'_0(r) = r^2 J_\phi^{(0)}[\psi_0(r)], \tag{12}\]
and an hierarchy of linear equations for the higher-order perturbations,
\[r^2 \psi''_{nk}(r) + r \psi'_{nk}(r) + [s(r) - k^2] \psi_{nk}(r) = b_{nk}(r), \quad \text{for} \quad s(r) = -r^2 J_\phi^{(0)}[\psi_0(r)], \tag{13}\]
where the source term \(b_{nk}(r)\) couples lower-order perturbations \(\psi_{nk}(r)\) only, with \(m < n\).

The major difficulty with this approach is that \(b_{nk}(r)\) also couples every flux derivative \(d^n p/d\psi^m\) and \(d^n Y/d\psi^m\), of order \(m < n\), evaluated at the zeroth-order equilibrium \(\psi_0(r)\), reflecting the assumption that \(p(\psi)\) and \(Y(\psi)\) are smooth functions of \(\psi\). In cases without TCR, these derivatives are readily computed applying the recursive operator
\[d^n/d\psi^m = \frac{1}{\psi_0} \frac{d}{dr} \left( \frac{d^{n-1}}{d\psi^{m-1}} \right) \tag{14}\]
as many times as needed [19]. However, when a PFR layer is present \(\psi_0\) vanishes there, requiring \((d/dr)(d^n/d\psi^m)\) to do the same for every order \(m\). This problem is circumvented providing input profiles, say \(J_\phi^{(0)}(\psi; a_1, \ldots, a_i)\) and \(p(\psi; b_1, \ldots, b_j)\), as analytical models depending on a few parameters \(a_1, \ldots, a_i\) and \(b_1, \ldots, b_j\) which are chosen to get a suitable solution to Eq. (12). Then, the desired flux derivatives immediately follow from the analytical models supplied.
Additionally, the input profiles should be split in two different branches, if some level of independence between the inner and the outer part of the equilibrium is intended. Indeed, because $\psi_0(r)$ is not monotonic, the equation $\psi_0(r) = \bar{\psi}$ may yield two distinct solutions, $r_a$ and $r_b$, with $r_a < r_L < r_b$. Hence, simple models like $J^0_\psi(\psi; a_1, \ldots, a_i)$ and $p(\psi; b_1, \ldots, b_j)$ do prescribe the same values, e.g.

$$p[\psi(r_a); b_1, \ldots, b_j] = p[\psi(r_b); b_1, \ldots, b_j],$$

for distinct flux surfaces located inside and outside the PFR layer. This gives rise to hollow pressure profiles and stiff $J^0_\psi$ distributions [12], which cannot be tailored to model more realistic cases [15].

As an example, consider the input profiles defined as

$$J^0_\psi(\psi) = \begin{cases} J_0 + \gamma^2 \psi^4 & \iff r \leq r_L \\ (a_0 + a_1 \Delta + a_2 \Delta^2 + a_3 \Delta^3 + \frac{\gamma}{2} e^{-10^4 \Delta^2}) e^{-20\Delta} & \iff r > r_L \end{cases}$$

(15a)

$$p(\psi) = \begin{cases} A & \iff r \leq r_L \\ (b_0 + b_1 \Delta + b_2 \Delta^2 + b_3 \Delta^3 + b_4 \Delta^4) e^{-30\Delta} & \iff r > r_L \end{cases}$$

(15b)

with $\Delta = \psi(r) - \psi(r_L)$. These are depicted in Fig. 5, for $J_0 = -3.1$ MA m$^{-2}$ and $A = 3 \times 10^4$ N m$^{-2}$, along with $\gamma = 300$ in order to get $I_p = 1.7$ MA, while the remaining coefficients are chosen to ensure flux-derivative continuity up to a suitable order. Remaining parameters are $a = 0.65$ m, $R_0 = 3.0$ m, and $B_0 = 2.75$ T. The corresponding flux-surface topology is displayed in Fig. 6, where the inner island system is clearly visible.
CONCLUSIONS

Some concepts borrowed from catastrophe theory were used to understand several properties of nested TCR equilibria. Namely, it was shown that the existence of a PFR layer matching a given flux surface is a sufficient condition for the resulting equilibrium to be degenerate and highly sensitive to perturbations in model parameters, being easily transformed into a non-nested configuration. More important, nested TCR equilibria were shown to be parametrized by a measure-zero set in parameter space, demanding them to be also a measure-zero set in the space of all GS solutions with TCR, meaning that they are not a representative class of solutions, as opposed to non-nested configurations. In addition, TCR equilibria computation was shown to demand special care when providing the zeroth-order input profiles needed to start a perturbative approach intended to solve the GS equation. Being these profiles necessarily flux functions, it was shown how splitting them in two branches allowed for some level of independence between equilibria inside and outside the PFR layer.
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